New additivity properties of the relative entropy of entanglement and its generalizations

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Resource monotones

- Here, a resource theory is specified by a convex set $\mathcal{F}$ of free states.
- Free operations are any/the maximal set of channels that preserve free states.

Example: resource theory of entanglement (separable states are free, local operations and classical communication are free operations)

- A function $\mathfrak{R}$ from states to $[0, +\infty]$ is an entanglement monotone if it does not increase under free operations.

Examples: relative entropy of entanglement, generalized robustness of entanglement, log-negativity...

The additivity question

A fundamental problem is to establish whether a certain monotone is tensor-additive.

We say that $\mathfrak{R}$ is *tensor-additive* for the states ρ_1 and ρ_2 if

$$\mathfrak{R}(\rho_1 \otimes \rho_2) = \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2)$$

- Is $\mathfrak{R}$ additive for any states ρ_1 and ρ_2?
- What are the minimum requirement on the states ρ_1 and ρ_2 to ensure additivity?

From monotones to necessary conditions for transformations

- When can a state ρ be transformed to σ using free operations?
- A necessary conditions is that $\mathfrak{R}(\rho) \geq \mathfrak{R}(\sigma)$ for all resource monotones.
- If we allow catalysts, i.e. free transformations of the form

$$\rho \otimes \tau \to \sigma \otimes \tau,$$

  then there might be fewer necessary conditions. Which survive?
- Certainly the additive monotones still need to be ordered, since

$$\begin{aligned} \mathfrak{R}(\rho) &= \mathfrak{R}(\rho \times \tau) - \mathfrak{R}(\tau) \\ &\geq \mathfrak{R}(\sigma \times \tau) - \mathfrak{R}(\tau) \\ &= \mathfrak{R}(\sigma) \end{aligned}$$

A concrete question

The resource theory of entanglement is well-understood when we restrict our attention to pure input and output states.

- The necessary **and sufficient** conditions for catalytic transformations are due to Klimesh and Turgut. With p and q the Schmidt vectors for input and output state and u the uniform vector, these are

$$D_\alpha(p\|u) \geq D_\alpha(q\|u) \quad \text{and} \quad D_\alpha(u\|p) \geq D_\alpha(u\|q)$$

  for all $\alpha \geq \frac{1}{2}$ where D_α are the classical Rényi divergences.
- But this argument implicitly assumes that catalysts are pure too!
- Are these conditions still necessary if we allow for mixed catalysts?
- We need additivity when just one of the two states is pure.

Non-Additivity of the entanglement monotones based on a quantum relative entropy

Non-additivity of entanglement monotones based on a quantum relative entropy

We define the monotone based on a quantum relative entropy

$$\min_{\sigma \in \mathsf{SEP}} \mathbb{D}(\rho \| \sigma)$$

Here, $\mathbb{D}$ satisfies [1]

- Data-processing inequality:
  $\mathbb{D}(\rho\|\sigma) \geq \mathbb{D}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma))$ for any quantum channel $\mathcal{E}$.
- Additivity under tensor products:
  $\mathbb{D}(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = \mathbb{D}(\rho_1\|\sigma_1) + \mathbb{D}(\sigma_2\|\sigma_2)$
- Normalization condition:
  $\mathbb{D}(|0\rangle\langle 0| \| \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|) = 1$

[1] Gour, Gilad, and Marco Tomamichel. "Optimal extensions of resource measures and their applications." Physical Review A 102, no. 6 (2020): 062401.

Non-additivity of entanglement monotones based on a quantum relative entropy

Theorem 1

Monotones based on relative entropies are not additive for general states.

For $d \gg 1$ we have that

$$\min_{\sigma \in \mathsf{SEP}} \mathbb{D}(\rho_- \otimes \rho_- \| \sigma) \sim \min_{\sigma \in \mathsf{SEP}} \mathbb{D}(\rho_- \| \sigma) \neq 2 \min_{\sigma \in \mathsf{SEP}} \mathbb{D}(\rho_- \| \sigma)$$

Here ρ_- is the bipartite antisymmetric (Werner) state.

Additivity properties of the relative entropy of entanglement

Additivity of the relative entropy of entanglement

The relative entropy of entanglement is

$$\mathfrak{D}(\rho) := \min_{\sigma \in \mathsf{SEP}} D(\rho \| \sigma)$$

with D the Umegaki relative entropy.

- A *maximally correlated state* has the form

$$\rho_{\mathsf{MC}} = \sum_{jk} \rho_{jk} |j, j\rangle\langle k, k| \,.$$

- A bipartite pure state is a maximally correlated state.

Theorem 2

Let ρ_1 be a maximally correlated state. Then, for any state ρ_2 we have

$$\mathfrak{D}(\rho_1 \otimes \rho_2) = \mathfrak{D}(\rho_1) + \mathfrak{D}(\rho_2) \,.$$

Additivity of the relative entropy of entanglement

Theorem 3

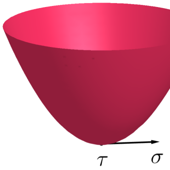
Let ρ_1 be a N-partite state. Moreover, let $\tau_1 \in \arg\min_{\sigma \in \mathsf{SEP}} D(\rho_1 \| \sigma)$. If $[\rho_1, \tau_1] = 0$ and $\rho_1 \tau_1^{-1}$ is non-negative, then for any N-partite state ρ_2 we have that

$$\mathfrak{D}(\rho_1 \otimes \rho_2) = \mathfrak{D}(\rho_1) + \mathfrak{D}(\rho_2)\,.$$

This class includes the separable, Bell diagonal, generalized Dicke and the isotropic states.

Main idea

Check necessary and sufficient conditions for minimum of convex function.



Theorem 4

Let ρ be a quantum state. Then $\tau \in \arg\min_{\sigma \in SEP} D(\rho \| \sigma)$ if and only if $\mathrm{Tr}(\sigma\, \Xi(\rho, \tau)) \leq 1$ for all $\sigma \in SEP$ where

$$\Xi(\rho, \tau) = \int_0^\infty (\tau + t)^{-1} \rho (\tau + t)^{-1} \mathrm{d}t$$

$\rightarrow$ We essentially want to show, using properties of τ_1 and τ_2,

$$\mathrm{Tr}(\sigma \Xi(\rho_1 \otimes \rho_2, \tau_1 \otimes \tau_2)) \leq 1 \quad \forall \sigma \in \mathrm{SEP}\,.$$

Generalizations

α-z Rényi relative entropy of entanglement

The α-z Rényi relative entropy provides a general framework to address different families of quantum Rényi divergences.

Let $\alpha \in (0,1) \cup (1,\infty)$, $z > 0$ and $\rho, \sigma \in \mathcal{S}(A)$ with $\rho \neq 0$. Then the α-z *Rényi relative entropy* of σ with ρ is defined as [2]

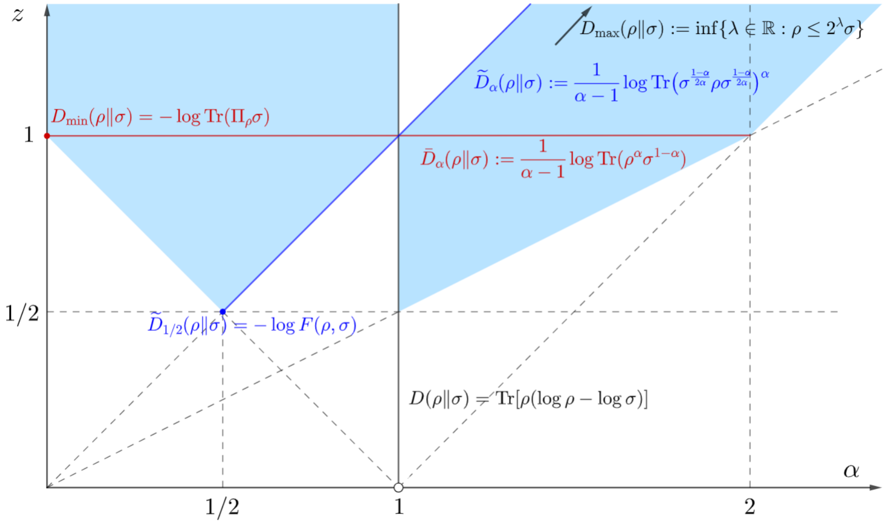
$$D_{\alpha,z}(\rho\|\sigma) := \begin{cases} \frac{1}{\alpha-1} \log \mathrm{Tr}\left(\rho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \rho^{\frac{\alpha}{2z}}\right)^z & \text{if } (\alpha < 1 \wedge \rho \not\perp \sigma) \vee \rho \ll \sigma \\ +\infty & \text{else} \end{cases} .$$

The α-z *Rényi relative entropy of entanglement* is

$$\mathfrak{D}_{\alpha,z}(\rho) := \min_{\sigma \in \mathrm{SEP}} D_{\alpha,z}(\rho\|\sigma) .$$

[2] Audenaert, Koenraad MR, and Nilanjana Datta. 'α-z-Rényi relative entropies.' Journal of Mathematical Physics 56, no. 2 (2015): 022202.

α-z Rényi relative entropy



Some well-known points

- $\alpha = 1\ (z \neq 0)$: **Relative entropy of entanglement**
- $\alpha = z = \infty$: **Generalized (log) robustness of entanglement**
  The generalized robustness is given by

$$\mathfrak{R}_g(\rho) := \min\left\{ s \geq 0 : \exists \omega \in \mathcal{S}(A) \text{ s.t } \frac{1}{1+s}(\rho + s\omega) \in \text{SEP} \right\}.$$

  We have $\mathfrak{D}_{\infty,\infty}(\rho) = \log\left(1 + \mathfrak{R}_g(\rho)\right)$.
- $\alpha = z = 1/2$: **Geometric measure of entanglement**

$$E_G(|\psi\rangle) = 1 - \max_{|\phi\rangle \in \text{PRO}} |\langle\phi|\psi\rangle|^2$$
$$E_G(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E_G(|\psi_i\rangle).$$

  We have $\mathfrak{D}_{1/2,1/2}(\rho) = -\log F_s(\rho) = -\log\left(1 - E_G(\rho)\right)$ where $F_s(\rho) = \max_{\sigma \in \text{SEP}} F(\rho, \sigma)$ is the fidelity of separability.

Additivity of the α-z Rényi relative entropy of entanglement

Theorem 5

Let ρ_1 be a maximally correlated state and $(\alpha, z) \in \mathcal{D}$. Then, for any state ρ_2 we have

$$\mathfrak{D}_{\alpha,z}(\rho_1 \otimes \rho_2) = \mathfrak{D}_{\alpha,z}(\rho_1) + \mathfrak{D}_{\alpha,z}(\rho_2) \,.$$

Theorem 6

Let ρ_1 be a N-partite state and $(\alpha, z) \in \mathcal{D}$. Moreover, let $\tau_1 \in \arg\min_{\sigma \in \mathsf{SEP}} D_{\alpha,z}(\rho_1 \| \sigma)$. If $[\rho_1, \tau_1] = 0$ and $\rho_1^{\alpha} \tau_1^{-\alpha}$ is non-negative, then for any N-partite state ρ_2 we have that

$$\mathfrak{D}_{\alpha,z}(\rho_1 \otimes \rho_2) = \mathfrak{D}_{\alpha,z}(\rho_1) + \mathfrak{D}_{\alpha,z}(\rho_2) \,.$$

Applications to catalytic transformations of pure entangled states

Understand the power of mixed catalysts

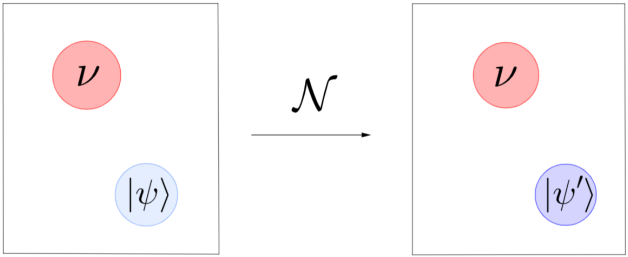
We have

$$\mathfrak{D}_{\alpha,z}(|\psi\rangle)+\mathfrak{D}_{\alpha,z}(\nu) = \mathfrak{D}_{\alpha,z}(|\psi\rangle\otimes\nu) \geq \mathfrak{D}_{\alpha,z}(|\psi'\rangle\otimes\nu) = \mathfrak{D}_{\alpha,z}(|\psi'\rangle)+\mathfrak{D}_{\alpha,z}(\nu)$$

This implies the following set of necessary conditions:

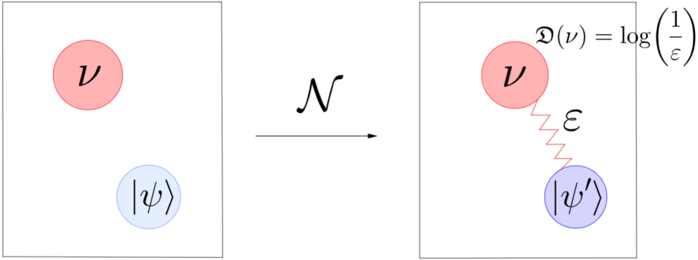
$$\mathfrak{D}_{\alpha,z}(|\psi\rangle) \geq \mathfrak{D}_{\alpha,z}(|\psi'\rangle) \qquad \forall(\alpha,z)\in\mathcal{D}$$

(Let $|\psi\rangle = \sum_i \sqrt{p_i}|i,i\rangle$. Then $\mathfrak{D}_{\alpha,z}(|\psi\rangle) = H_\beta(\vec{p})$ with $(1-\alpha)/z + 1/\beta = 1$.)



Find fundamental limits of correlated catalytic transformations

The additivity result for $\alpha = z \in [1/2, 1)$ is a key ingredient to prove that, for a large class of correlated catalytic transformations, the resource of the catalyst must diverge as the correlations vanish.[3]



[3] Rubboli, Roberto, and Marco Tomamichel. "Fundamental limits on correlated catalytic state transformations." Physical Review Letters 129, no. 12 (2022): 120506.

Thanks for your attention!

Sketch of the proof - REE - pure states

We can write any pure separable state in the partition $AA':BB'$ as $\sigma = \sum_i a_i |i, \phi_i\rangle_{AA'} \otimes \sum_j b_j |j, \psi_j\rangle_{BB'}$. We have

$$\mathrm{Tr}(\sigma \Xi(\rho_1 \otimes \rho_2, \tau_1 \otimes \tau_2)) \tag{1}$$

$$\leq \sum_{iji'j'} a_i b_j a_{i'} b_{j'} |\langle ij|\langle \phi_i \psi_j| \Xi(\rho_1 \otimes \rho_2, \tau_1 \otimes \tau_2) |i'j'\rangle |\phi_{i'} \psi_{j'}\rangle| \tag{2}$$

$$\leq \sum_{iji'j'} a_i b_j a_{i'} b_{j'} \Big(\langle ij|\langle \phi_i \psi_j| \, \Xi(\rho_1 \otimes \rho_2, \tau_1 \otimes \tau_2) \, |ij\rangle |\phi_i \psi_j\rangle \tag{3}$$

$$\times \langle i'j'|\langle \phi_{i'} \psi_{j'}| \, \Xi(\rho_1 \otimes \rho_2, \tau_1 \otimes \tau_2) \, |i'j'\rangle |\phi_{i'} \psi_{j'}\rangle \Big)^{\frac{1}{2}} . \tag{4}$$

We then write $\tau_1 = \sum_i p_i |ii\rangle\langle ii|$ and $\tau_2 = \sum_r s_r |\xi_r\rangle\langle \xi_r|$. We have

$$\langle ij|\langle \phi_i \psi_j| \, \Xi(\rho_1 \otimes \rho_2, \tau_1 \otimes \tau_2) \, |ij\rangle |\phi_i \psi_j\rangle \tag{5}$$

$$= \delta_{ij} \sum_{r_1 r_2} \int_0^\infty (p_i s_1 + t)^{-1} (p_i s_2 + t)^{-1} p_i \langle \phi_i \psi_j | \xi_{r_1}\rangle \langle \xi_{r_1} | \rho_2 | \xi_{r_2}\rangle \langle \xi_{r_2} | \phi_i \psi_j\rangle \tag{6}$$

Sketch of the proof

Therefore we have

$$\langle ij|\langle\phi_i\psi_j|\,\Xi(\rho_1\otimes\rho_2,\tau_1\otimes\tau_2)\,|ij\rangle|\phi_i\psi_j\rangle = \delta_{ij}\langle\phi_i\psi_j|\Xi(\rho_2,\tau_2)|\phi_i\psi_j\rangle \leq \delta_{ij} \tag{8}$$

This implies that

$$\mathrm{Tr}(\sigma\Xi(\rho_1\otimes\rho_2,\tau_1\otimes\tau_2)) \leq \sum_{iii'i'} a_i b_i a_{i'} b_{i'} \leq 1 \qquad \forall\sigma\in\mathsf{SEP} \tag{9}$$

which is what we wanted to prove.

Some states

STATES	VALUE OF $\mathfrak{D}_{\alpha,z}$
Bipartite pure $\lvert\rho(\vec{p})\rangle = \sum_i \sqrt{p_i}\lvert i,i\rangle$	$H_\beta(\vec{p})$ where $\frac{1-\alpha}{z}+\frac{1}{\beta}=1$
Bell diagonal $\rho_{\mathrm{BD}}(\vec{\lambda}) = \sum_{j=1}^{4}\lambda_j\lvert\psi_j\rangle\langle\psi_j\rvert$	$\begin{cases} 0 & \text{if } \lambda_{\max}\in[0,\frac{1}{2}] \\ 1-H_\alpha(\lambda_{\max},1-\lambda_{\max}) & \text{if } \lambda_{\max}\in[\frac{1}{2},1] \end{cases}$
Generalized Werner $\rho_W(p) = p\frac{2}{d(d+1)}P_{AB}^{\mathrm{SYM}} + (1-p)\frac{2}{d(d-1)}P_{AB}^{\mathrm{AS}}$	$\begin{cases} 1-H_\alpha(p,1-p), & p\in[0,1/2] \\ 0 & p\in[1/2,1] \end{cases}$
Isotropic $\rho_{iso}(F) = \frac{1-F}{d^2-1}(\mathbb{1}-\lvert\Phi^+\rangle\langle\Phi^+\rvert) + F\lvert\Phi^+\rangle\langle\Phi^+\rvert$	$\begin{cases} \log d - H_\alpha\left(\frac{1-F}{(d-1)^{\frac{\alpha-1}{\alpha}}},F\right), & F\in[\frac{1}{d},1] \\ 0 & F\in[0,\frac{1}{d}] \end{cases}$
Generalized Dicke $S(\lvert N,\vec{k}\rangle) = \frac{1}{\sqrt{C_{n,\vec{k}}}}\sum_{\mathrm{P}} P\lvert \underbrace{0,\dots,0}_{k_0},\underbrace{1,\dots,1}_{k_1},\dots,\underbrace{d-1,\dots,d-1}_{k_d-1}\rangle$	$-\log\left(\frac{N!}{\prod_{j=0}^{d-1}k_j!}\prod_{j=0}^{d-1}\left(\frac{k_j}{N}\right)^{k_j}\right)$
Maximally correlated Bell diagonal $\rho_{\mathrm{MCBD}}(\vec{p}) = \sum_{k=0}^{d-1}p_k\lvert\psi_k\rangle\langle\psi_k\rvert$	$\log d - H_\alpha(\vec{p})$