Correlation in Catalysts Enables Arbitrary Manipulation of Quantum Coherence

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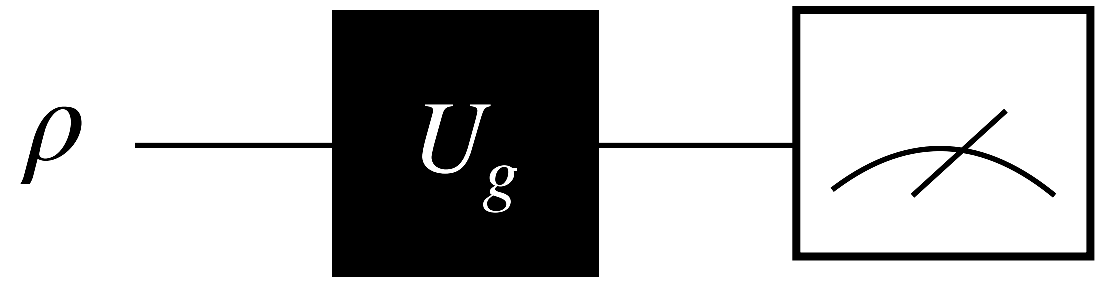
Resource Theory of Asymmetry

Symmetry group G with representation U_g for $g \in G$.

A state ρ is *symmetric* if $U_g \rho U_g^\dagger = \rho,\ \forall g \in G$. Otherwise, ρ is *asymmetric*.

Operational resource for metrology



- Quantify the amount of symmetry violation
- Characterize the possible state transformation with operations that respect symmetry

Covariant operations: $U_g \mathscr{E}(\rho) U_g^\dagger = \mathscr{E}(U_g \rho U_g^\dagger), \forall g \in G, \forall \rho$

Resource theory of asymmetry $\mathbb{F}$: symmetric states and $\mathbb{O}$: Covariant operations

Quantum Coherence as Asymmetry

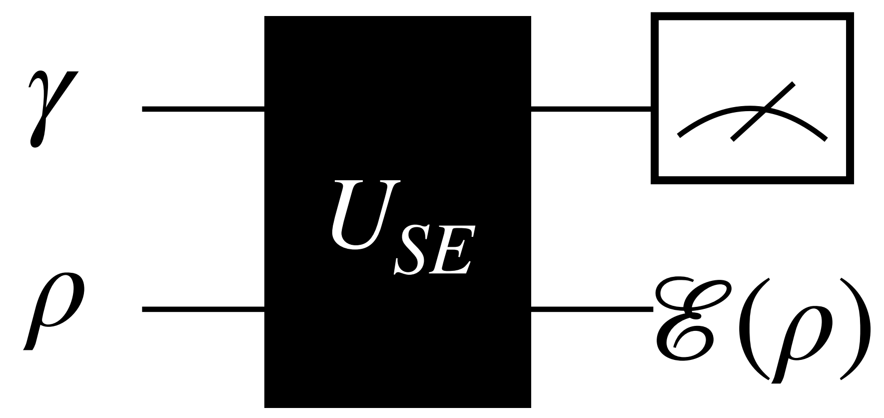
Consider a specific group $G = \mathrm{U}(1)$ with representation $U_t = e^{iHt}$. $H = \sum_i E_i |i\rangle\langle i|$

Asymmetric state has energetic coherence. e.g., $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- Resource for quantum clock
- "Quantum part" of quantum thermodynamics

Thermal Operations

$$\mathcal{E}(\rho) = \mathrm{Tr}_E\left(U_{SE}(\rho \otimes \gamma)\, U_{SE}^\dagger\right) \quad [U_{SE}, H_S + H_E] = 0 \quad \gamma = e^{-\beta H_E}/\mathrm{Tr}(e^{-\beta H})$$



- Thermal Operations are subclass of Covariant Operations with energy constraint.
- Free energy can be decomposed into classical and quantum parts.

$$\beta[F(\rho) - F(\gamma)] = S(\rho\|\gamma) = \underbrace{S(\Pi(\rho)\|\gamma)}_{\text{classical}} + \underbrace{C(\rho)}_{\text{quantum}}$$

$\Pi(\rho)$: dephasing with energy eigenabsis

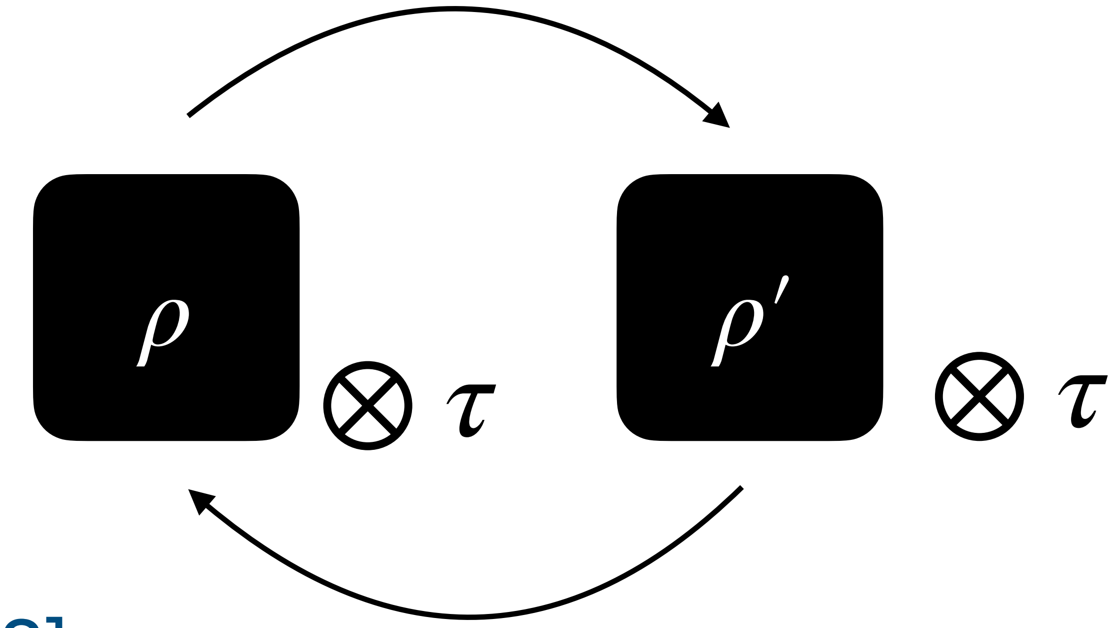
$C(\rho)$: relative entropy of coherence [Lostaglio et al. PRX '15]

Resource Manipulation with Catalysts

Ultimate transformation capability: One can consider using the help of *catalysts.*

Product catalysts

For a resource theory with free operations $\mathbb{O}$, $\rho \otimes \tau_C \underset{\mathbb{O}}{\to} \rho' \otimes \tau_C$ $\quad \tau_C$: catalyst



e.g.) Quantum thermodynamics (ρ, ρ': block-diagonal)

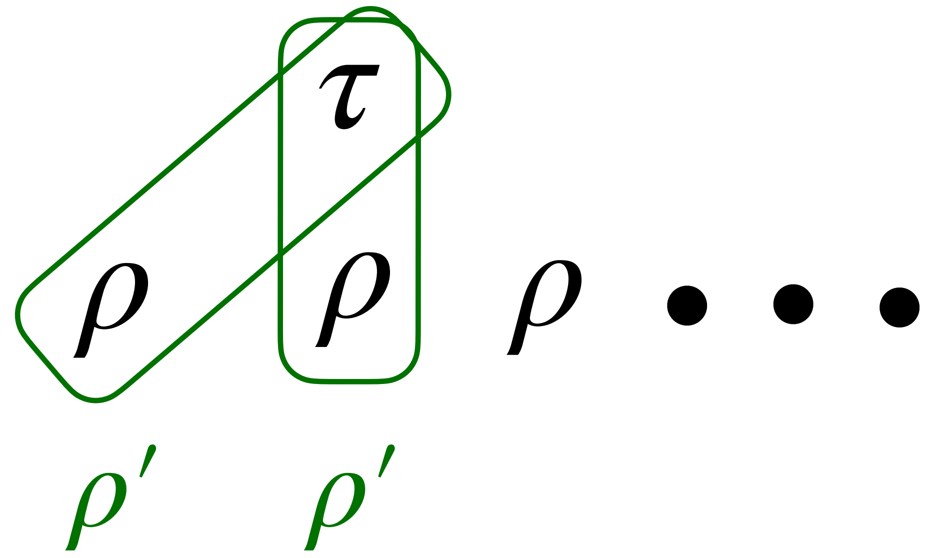
$\rho \xrightarrow[\text{Thermal}]{} \rho'$: thermo-majorization [Horodecki, Oppeheim, Nat. Comm. '13]

$\rho \otimes \tau_C \xrightarrow[\text{Thermal}]{} \rho' \otimes \tau_C$: "second laws". $F_\alpha(\rho) \geq F_\alpha(\rho'), \forall \alpha$ [Brandao et al., PNAS, '15]

Correlated Catalysts

$$\rho \otimes \tau_C \to \rho'_{SC} \text{ such that } \operatorname{Tr}_C \rho'_{SC} = \rho' \text{ and } \operatorname{Tr}_S \rho'_{SC} = \tau_C$$

Correlation between system and catalyst



The catalyst can be reused indefinitely as long as input states are freshly prepared.

e.g.) Quantum thermodynamics with correlated catalysts

$F(\rho) \geq F(\rho')$ with (1) thermal operations for block-diagonal ρ, ρ' [Müller, PRX, '18]

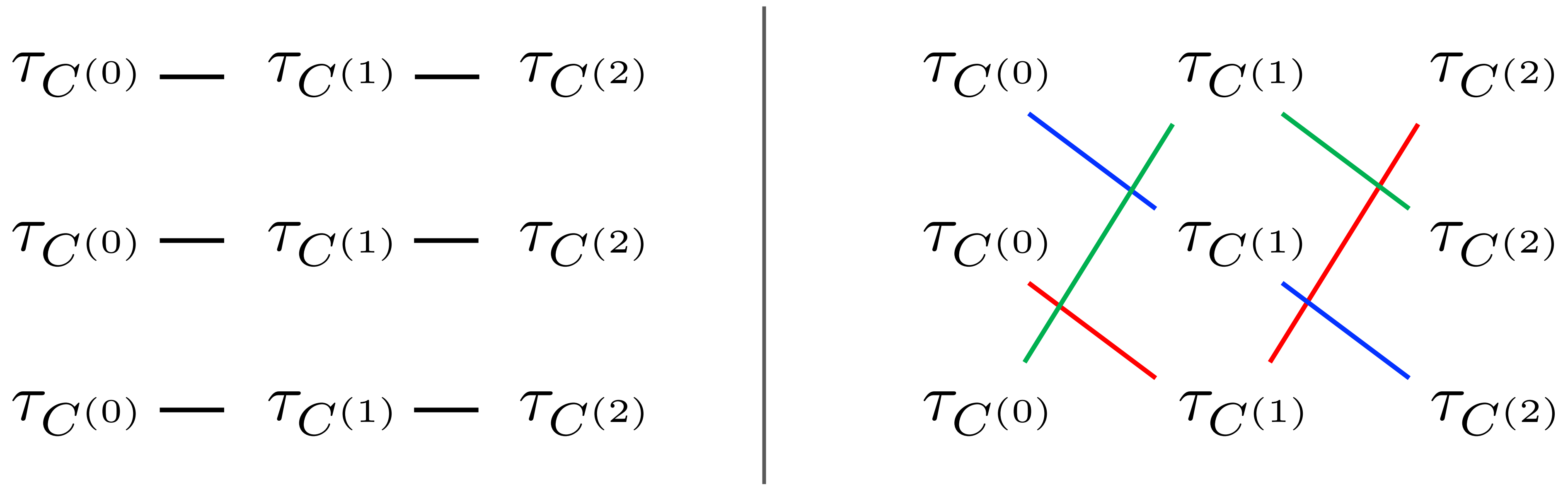
(2) Gibbs-preserving operations for general ρ, ρ' [Shiraishi, Sagawa, PRL, '21]

Marginal Catalysts

$$\rho \otimes \tau_{C^{(0)}} \cdots \otimes \tau_{C^{(K-1)}} \to \rho' \otimes \tau_{C^{(0)} \dots C^{(K-1)}}$$

$$\mathrm{Tr}_{\overline{C^{(j)}}} \left(\tau_{C^{(0)} \dots C^{(K-1)}} \right) = \tau_{C^{(j)}}, \ \forall j$$

Correlation between multiple catalysts



Infinite repeatability is lost due to correlation.

- Catalyst is partially reusable.
- Theoretical understanding of different operational setting.

<u>Previously considered in quantum thermodynamics</u>

$F(\rho) \geq F(\rho')$ with thermal operations for block-diagonal ρ and ρ'. [Lostaglio et al., PRL '15]

	Q. Thermo	Coherence
Product	$F_\alpha(\rho) \geq F_\alpha(\rho'), \forall\alpha$ (block diagonal)	
Correlated	$F(\rho) \geq F(\rho')$ (block diagonal) (general, Gibbs-preserving)	
Marginal	$F(\rho) \geq F(\rho')$ (block diagonal)	

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Correlated	$F(\rho) \geq F(\rho')$ (block diagonal) (general, Gibbs-preserving)	Coherence no-broadcasting [Marvian, Spekkens, PRL ’19] [Lostaglio, Muller, PRL ’19]
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Marginal	$F(\rho) \geq F(\rho')$ (block diagonal) (general, Gibbs-preserving)	No restriction (This work)

Arbitrary Manipulation is Possible

$$\rho \otimes \tau_{C^{(0)}} \cdots \otimes \tau_{C^{(K-1)}} \to \rho' \otimes \tau_{C^{(0)} \ldots C^{(K-1)}}$$

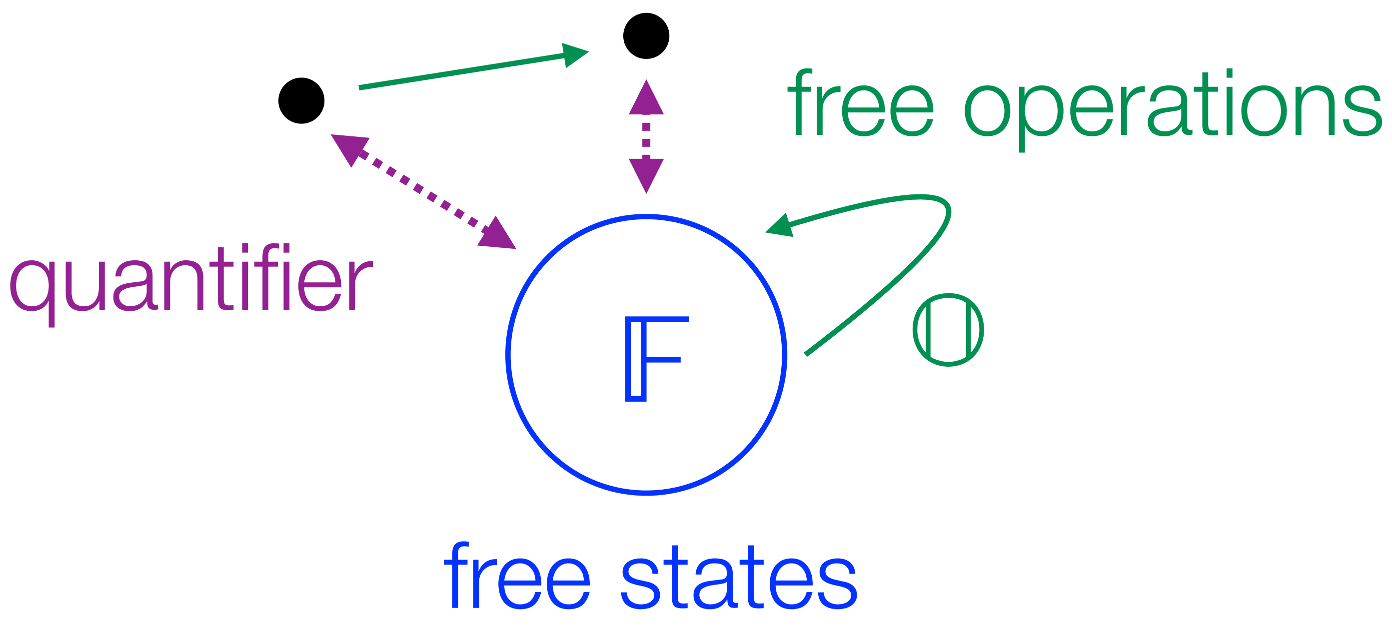
$$\mathrm{Tr}_{\overline{C^{(j)}}} \left(\tau_{C^{(0)} \ldots C^{(K-1)}} \right) = \tau_{C^{(j)}}, \ \forall j$$

Main result

For arbitrary states ρ and ρ', ρ can be transformed to ρ' by a marginal-catalytic covariant transformation with arbitrarily small error.

- ρ can even be an incoherent state.
- Unlike the case of thermodynamics, coherence transformation has no restriction.
- Can be regarded as a new type of embezzlement, but with a very different mechanism.

Restriction of Marginal Catalysts



- $\mathfrak{R}(\rho_{12}) \geq \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2),\ \forall \rho_{12}$ superadditivity
- $\mathfrak{R}(\rho_1 \otimes \rho_2) = \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2),\ \forall \rho_1, \rho_2$ product additivity

If ρ can be transformed to ρ' by a marginal-catalytic or correlated-catalytic free transformation, then every super additive, product-additive resource measure satisfies $\mathfrak{R}(\rho) \geq \mathfrak{R}(\rho')$

If there exists even a single superadditive, product-additive, and faithful resource measure, arbitrary resource transformation is impossible.

e.g., quantum thermo (free energy), speakable coherence (rel. ent. coherence), entanglement (squashed ent.)

Achievable Catalytic Transformation

We discussed necessary conditions for catalytic transformation. **What about sufficiency?**

Asymptotic transformation can be converted to single-shot catalytic transformation.

c.f. [Shiraishi, Sagawa, PRL, '21]

If ρ can be transformed to ρ' by an asymptotic free transformation, then ρ can be transformed to ρ' by a free transformation with correlated and marginal catalysts.

Completely characterize the correlated and marginal catalytic transformation of

- Quantum thermodynamics with Gibbs-preserving operations [Shiraishi, Sagawa, PRL '21]
- LOCC pure state transformation [Kondra et al., PRL '21] [Kipka-Bartosik, Skrzypczyk, PRL '21]
- Speakable coherence distillation

Operational meaning of relative entropy measures in terms of single-shot transformation.

Coherence Manipulation with Correlated Catalysts

$\rho \otimes \tau_C \to \rho'_{SC}$ such that $\mathrm{Tr}_C\, \rho'_{SC} = \rho'$ and $\mathrm{Tr}_S\, \rho'_{SC} = \tau_C$

Can we also realize an arbitrary coherence transformation with correlated catalysts?

We at least need some initial coherence due to the coherence no-broadcasting theorem.

Conjecture

If all energy differences for nonzero off-diagonals of ρ' can be expressed as a linear combination of those of ρ, then ρ can be transformed to ρ' by a correlated-catalytic covariant operation with an arbitrarily small error.

We can show a weaker version of this statement with a broader class of operations.

Could be a key toward the complete characterization of the capability of **thermal operations with correlated catalysts.**

Summary

- Marginal-catalytic covariant operation can realize arbitrary state transformation.
- This is possible by a peculiar property of coherence measure. We derived a general restriction on such an anomalous transformation valid for general resource theories.
- Presented achievable condition in relation to asymptotic transformation and precise characterization for some theories in combination with the above restriction.
- Showed that quasi-correlated-catalytic covariant transformation allows for a wide class of state transformation.

Outlook

- Prove or disprove the conjecture on correlated-catalytic covariant transformation.
- Characterization of correlated-catalytic and marginal-catalytic thermal transformations.
    - Can we show that if there is nonzero coherence in the initial state, then transformation rule is governed by the free energy?
    - Can we extend the result for semiclassical transformation with marginal catalysts to a quantum setting?
- Extension to general groups.

Thank you!