Symmetric distinguishability as a quantum resource

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Joint work with: Nilanjana Datta, Gilad Gour, Xin Wang and Mark M. Wilde

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Quantum resources:
From mathematical foundations to operational characterisation
Singapore, 6 December 2022

Introduction

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- Fundamental objects involved are **elementary quantum sources**. We study transformations between named sources via **free operations**.
- Key result: The **quantum Chernoff divergence** is fundamental exchange rate in RTSD.

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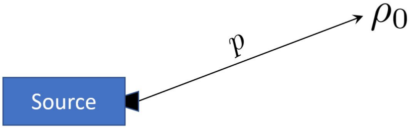

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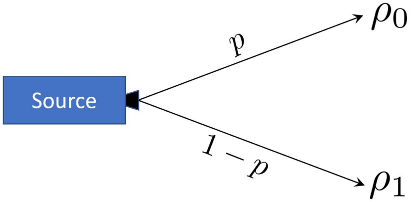

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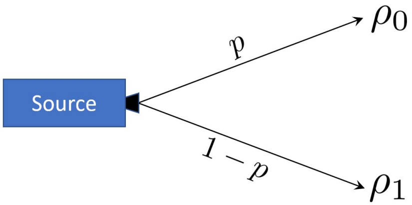

Such sources can be represented by classical quantum (c-q) state

$$\rho_{XA} := p|0\rangle\langle 0| \otimes \rho_0 + (1-p)|1\rangle\langle 1| \otimes \rho_1$$

Symmetric Hypothesis testing

Task: Given quantum source $\rho_{XA} \equiv (p, \rho_0, \rho_1)$, discriminate between the states ρ_0 and ρ_1.

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Infinite resource: ρ_0 and ρ_1 perfectly distinguishable ($\rho_0 \perp \rho_1$) and hence $\mathrm{SD}(\rho_{XA}) = \infty$.

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Symmetric Hypothesis testing: Asymptotic i.i.d. setting

Source emitting n copies of quantum states represented by $\rho_{XA}^{(n)} \equiv \left(p, \rho_0^{\otimes n}, \rho_1^{\otimes n}\right)$.

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Hence, $\rho_{XA}^{(n)}$ becomes an infinite resource as $n \to \infty$.

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Resource theory of symmetric distinguishability

Free operations for transformation between quantum sources

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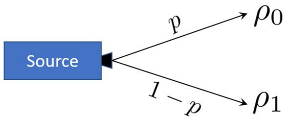

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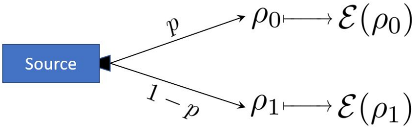

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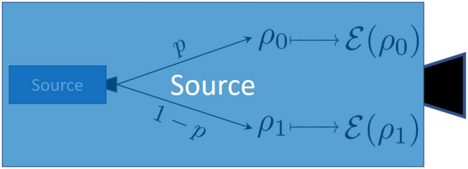

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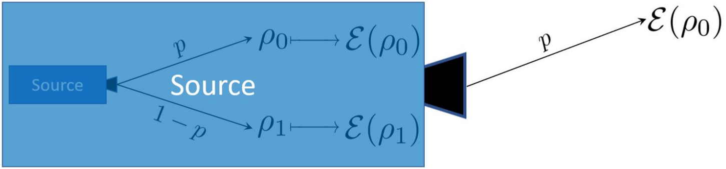

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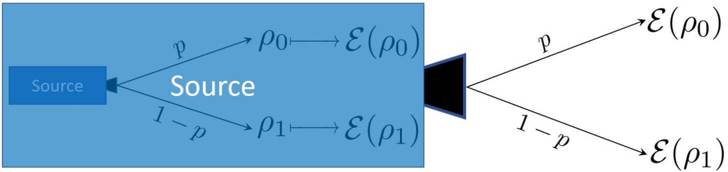

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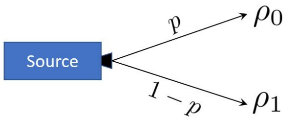

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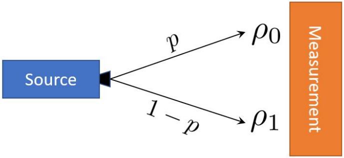

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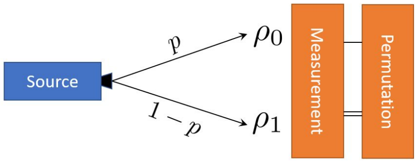

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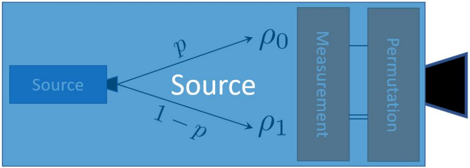

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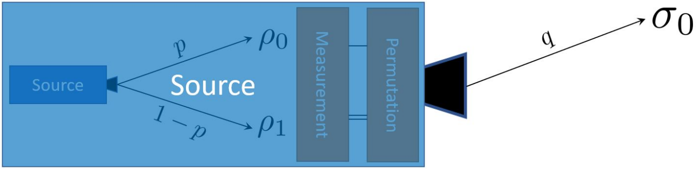

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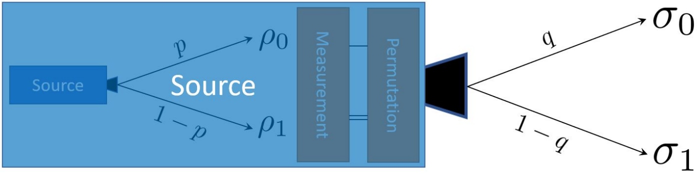

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Monotonicity of minimum error probability:

$$p_{\mathrm{err}}(\mathcal{N}(\rho_{XA})) \geq p_{\mathrm{err}}(\rho_{XA}), \text{ for all } \mathcal{N} \in \mathrm{CDS}.$$

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Error measure:

We allow for errors in the transformations which will be measured by the **scaled trace distance**

$$D'\left(\widetilde{\sigma}_{XB}, \sigma_{XB}\right) := \frac{\|\widetilde{\sigma}_{XB} - \sigma_{XB}\|_1}{2p_{\mathrm{err}}(\sigma_{XB})}.$$

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Important properties:

- **Monotonic** under free operations:

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- **Key inequality** for proving converses in RTSD:

$$p_{\mathrm{err}}(\widetilde{\sigma}_{XB}) \leq \big(D'\left(\widetilde{\sigma}_{XB}, \sigma_{XB}\right) + 1\big)p_{\mathrm{err}}(\sigma_{XB}).$$

Asymptotic transformations

What it is the optimal (i.e. largest) achievable rate $\frac{m}{n}$ such that

$$\rho_{XA}^{(n)} \equiv (p, \rho_0^{\otimes n}, \rho_1^{\otimes n}) \longmapsto \sigma_{XB}^{(m)} \equiv (q, \sigma_0^{\otimes m}, \sigma_1^{\otimes m}) \qquad (1)$$

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We will denote by

$$R(\rho_{XA} \mapsto \sigma_{XB}), \quad \text{under free operations being } \mathrm{CPTP}_A,$$
$$R^\star(\rho_{XA} \mapsto \sigma_{XB}), \quad \text{under free operations being } \mathrm{CDS},$$

the **optimal asymptotic rate** of the transformation (1) such that the D'-error vanishes as $n \to \infty$.

Theorem
(Optimal asymptotic rate of transformations in RTSD)

Under free operations being CDS :

$$R^\star(\rho_{XA} \mapsto \sigma_{XB}) = \frac{\xi(\rho_0, \rho_1)}{\xi(\sigma_0, \sigma_1)}.$$

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- The result shows that the RTSD is asymptotically reversible.
- The strong converse property holds: Any transformation with rate larger than $\frac{\xi(\rho_0,\rho_1)}{\xi(\sigma_0,\sigma_1)}$ leads to infinite D'-error as $n \to \infty$.

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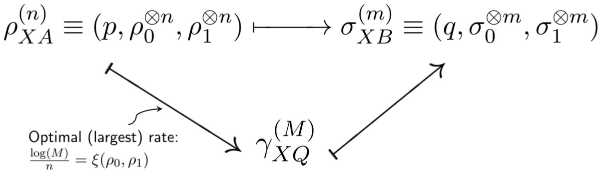

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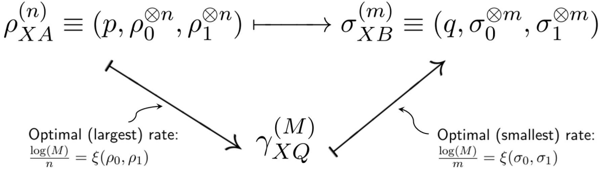

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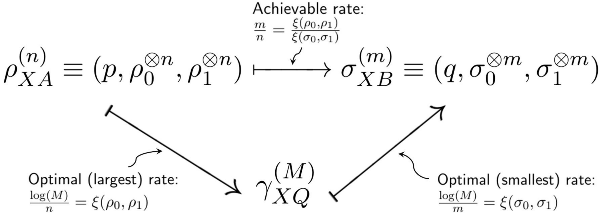

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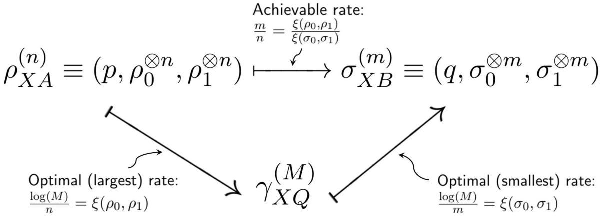

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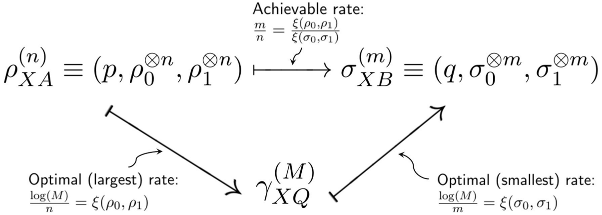

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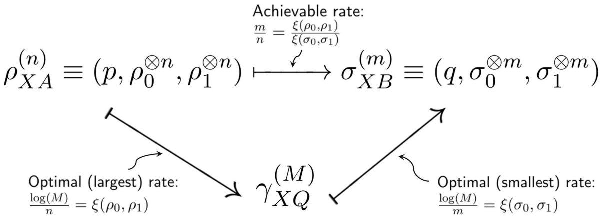

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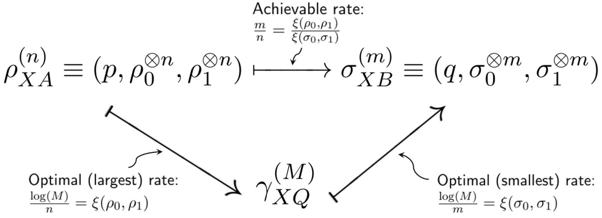

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$$\implies \frac{\xi(\rho_0,\rho_1)}{\xi(\sigma_0,\sigma_1)} \geq \frac{m}{n} + o(1) \quad \text{as } n \to \infty.$$

Definition (Golden unit)

For $M \in [1, \infty]$ and $q \in (0, 1)$ consider the (M, q)-golden unit

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For M large enough, $\gamma_{XQ}^{(M,q)}$ has well-behaved SD as

$$p_{\mathrm{err}}\left(\gamma_{XQ}^{(M,q)}\right) = \frac{1}{2M} \quad \text{and hence} \quad \mathrm{SD}\left(\gamma_{XQ}^{(M,q)}\right) = \log(M).$$

SD-distillation

Distil golden unit from initial source ρ_{XA} via free operations:

$$\rho_{XA} \equiv \big(p, \rho_0, \rho_1\big) \longmapsto \gamma_{XQ}^{(M,q)} \equiv (q, \pi_M, \sigma^x \pi_M \sigma^x) \, .$$

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One-shot exact distillable SD under free operations being CPTP_A

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$$Q_{\min}(\rho_0,\rho_1) = 2\min_{0\leq\Lambda\leq\mathbb{1}}\left\{\mathrm{Tr}(\Lambda\rho_1)\middle|\mathrm{Tr}((\mathbb{1}-\Lambda)\rho_0)\leq\mathrm{Tr}(\Lambda\rho_1)\right\},$$

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Asymptotic SD-distillation

Denoting again $\rho_{XA}^{(n)} \equiv (p, \rho_0^{\otimes n}, \rho_1^{\otimes n})$ we have:

Theorem

Optimal asymptotic rates of exact and approximate SD-distilation (for all $\varepsilon > 0$) are given by quantum Chernoff divergence:

$$\lim_{n\to\infty} \frac{\xi_d(\rho_{XA}^{(n)})}{n} = \lim_{n\to\infty} \frac{\xi_d^{\star}(\rho_{XA}^{(n)})}{n} = \lim_{n\to\infty} \frac{\xi_d^{\varepsilon}(\rho_{XA}^{(n)})}{n} = \lim_{n\to\infty} \frac{\xi_d^{\star,\varepsilon}(\rho_{XA}^{(n)})}{n}$$

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Here, $d_T(\rho_0, \rho_1) := \max\left\{D_{\max}(\rho_0\|\rho_1), D_{\max}(\rho_1\|\rho_0)\right\}$,[6] and $D_{\max}(\rho_0\|\rho_1) := \inf\left\{\lambda \middle| \rho_0 \le 2^\lambda \rho_1\right\}$[7].

[6] A.C. Thompson, Proc. Amer. Math. Soc., 14(3):438–443, 1963.
[7] N. Datta, IEEE Trans. on Inf. Theo., 55(6):2816–2826, 2009.

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[8] K. Li, The Ann. of Stat., 44(4):1661–1679, 2016.

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Thanks for your attention!

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