Fundamental Limits on Correlated Catalytic State Transformation

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Outline

1. Introduction to catalytic resource theories
    - resource theories
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2. Main result
   - Resource monotones
   - Necessary and sufficient conditions for correlated catalytic transformations
   - Main theorem
   - Additivity theorems in resource theories

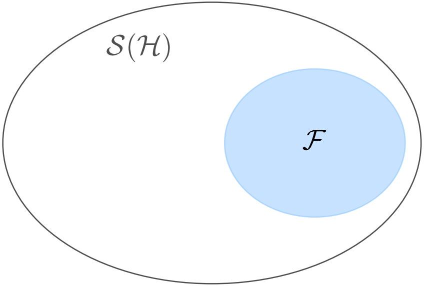
Outline

Introduction to catalytic resource theories

Resource theories

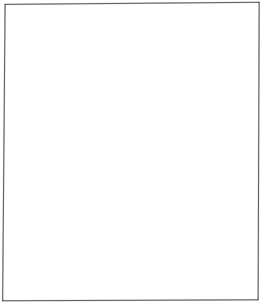
A quantum resource theory is defined by

- a set of free states $\mathcal{F}$
- a set of free operations with the property that free operations are closed under composition and map free states into free states.



Example: Entanglement theory. Free states are the separable states and free operations are local operations and classical communications (LOCC).

Free operations $\mathcal{FO}$



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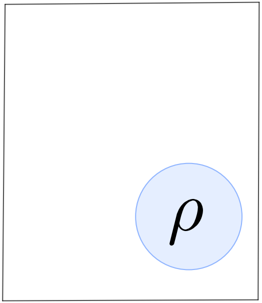

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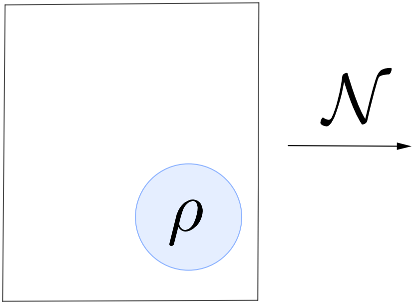

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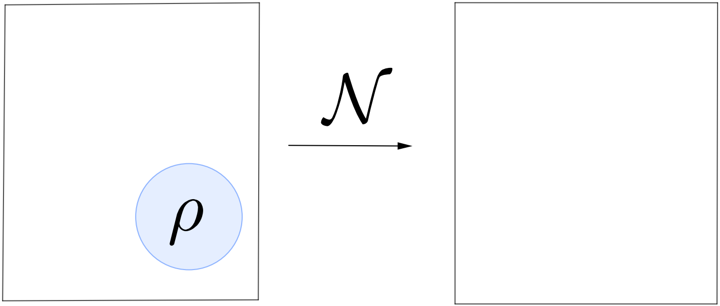

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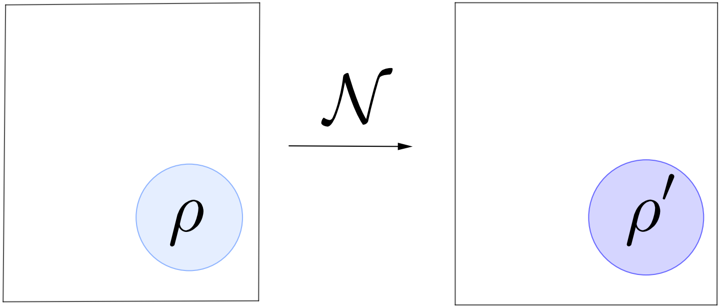
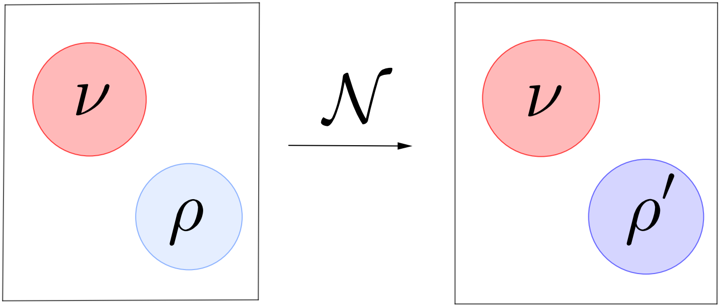


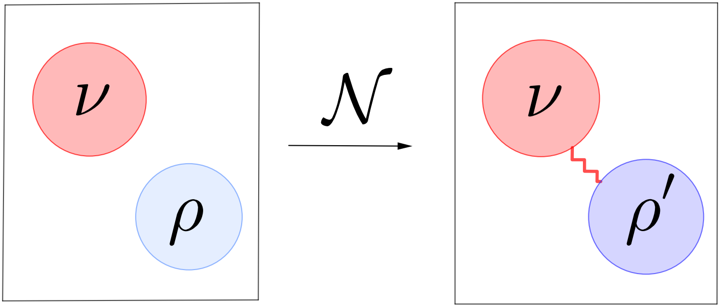
$\mathcal{N}(\rho) = \rho'$ and $\mathcal{N}$ is a free operation

Catalytic operations $\mathcal{CO}$



$\mathcal{N}(\rho \otimes \nu) = \rho' \otimes \nu$ and $\mathcal{N}$ is a free operation

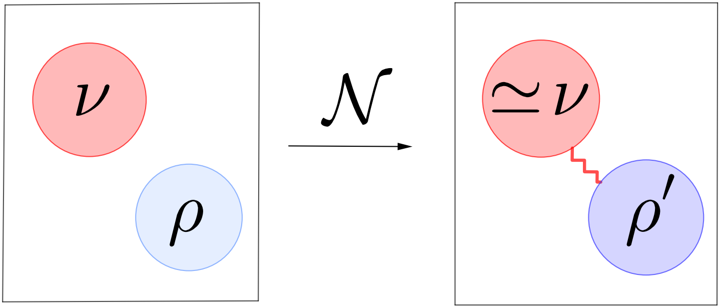
Correlated catalytic operations $\mathcal{CCO}$



$\mathcal{N}(\rho \otimes \nu) = \tau,\ \tau$ has marginals ρ' and ν and $\mathcal{N}$ is a free operation.[1]

[1] Gallego, Rodrigo, Jens Eisert, and Henrik Wilming. "Thermodynamic work from operational principles." New Journal of Physics 18, no. 10 (2016): 103017.

Embezzlement



The catalyst is not recovered exactly.

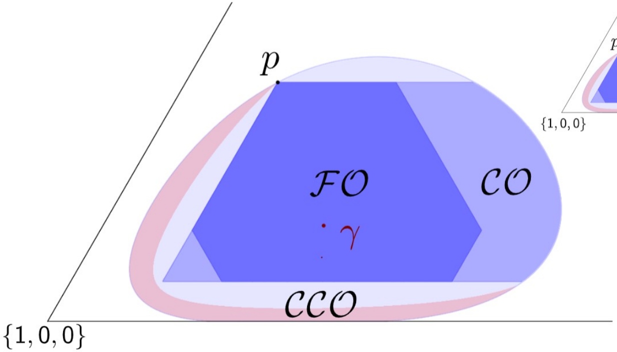
An exmaple: Catalytic transformations in classical resource theory of athermality

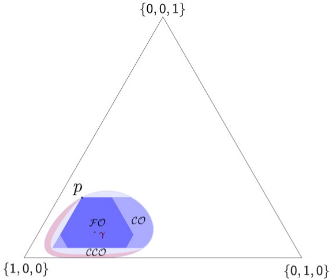
Resource theory of athermality.

- In resource theory of athermality the thermal state $\gamma = e^{-\beta H}/Z$ is the only free state in the theory.
- Free operations are Gibbs preserving maps, i.e. maps Λ such that $\Lambda(\gamma) = \gamma$.

An exmaple: Catalytic transformations in classical resource theory of athermality

In general we do not know the necessary and sufficient conditions for ρ' to be in $\mathcal{FO}(\rho)/\mathcal{CO}(\rho)/\mathcal{CCO}(\rho)$. But for classical resource theory of athermality..





Main result

The model

Definition

We say that ρ can be transformed into ρ' by an ε-correlated catalytic transformation if there exists a free operation $\mathcal{N}$ and a catalyst state ν such that

$$\mathcal{N}(\rho \otimes \nu) = \tau, \mathrm{Tr}_S[\tau] = \nu \text{ and } P(\rho' \otimes \nu, \tau) \leq \varepsilon \tag{1}$$

If this holds for any $\varepsilon > 0$ we say that ρ is transformable into ρ' by a correlated catalytic transformation.

Purified distance: $P(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$ where $F(\rho, \sigma) = (\mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|)^2$

Sandwiched Rényi divergences

Let $\alpha \in \left[\frac{1}{2}, 1\right) \cup (1, \infty)$ and $\rho, \sigma \in \mathcal{S}(A)$ with $\rho \neq 0$. Then the *sandwiched Rényi divergence* of σ with ρ is defined as

$$\widetilde{D}_\alpha(\rho\|\sigma) := \begin{cases} \frac{1}{\alpha-1} \log \mathrm{Tr}\left(\rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1}{2}}\right)^\alpha & \text{if } (\alpha < 1 \wedge \rho \not\perp \sigma) \vee \rho \ll \sigma \\ +\infty & \text{else} \end{cases} . \tag{2}$$

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In the limit $\alpha \to 1$ it converges to the Umegaki relative entropy

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}[\rho(\log \rho - \log \sigma)] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \\ +\infty & \text{else} \end{cases} \tag{3}$$

Resource monotones based on Rényi divergences - ‘catalytic monotones’

We say that a function $\mathfrak{R} : \mathcal{S}(A) \to [0, +\infty]$ is a resource monotone if it does not increase under free operations, i.e., if

$$\mathfrak{R}(\rho) \geq \mathfrak{R}(\mathcal{E}(\rho)) \tag{4}$$

for any state ρ and LOCC operation $\mathcal{E}$.

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We define the quantity for $\alpha \in \left[\frac{1}{2}, \infty\right]$

$$\mathfrak{D}_\alpha(\rho) = \inf_{\sigma \in \mathcal{F}} \widetilde{D}_\alpha(\rho \| \sigma) \tag{5}$$

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$$\mathfrak{D}_\alpha(\rho) = \inf_{\sigma \in \mathcal{F}} \widetilde{D}_\alpha(\rho \| \sigma) \tag{5}$$

We denote $\mathfrak{D}(\rho) := \inf_{\sigma \in \mathcal{F}} D(\rho \| \sigma)$. (special role in $\overline{\mathcal{CCO}}$)
Since the underling divergence $\widetilde{D}_\alpha$ satisfies the DPI, the above quantity does not increase under free operations, i.e. it is a resource monotone.

Build necessary and sufficient conditions for $\overline{\mathcal{CCO}}$

We discuss the implication of our result in

- entanglement theory (pure input and output states)
- coherence theory (pure output states)
- resource theory of athermality

[2] Winter, A., Yang, D. (2016). Physical review letters, 116(12), 120404.

[3] Shiraishi, N. and Sagawa, T. (2021). Physical Review Letters, 126(15), 150502.

[4] Kondra, Tulja Varun, Chandan Datta, and Alexander Streltsov. Physical Review Letters 127, no. 15 (2021): 150503.

Build necessary and sufficient conditions for $\overline{\mathcal{CCO}}$

We discuss the implication of our result in

- entanglement theory (pure input and output states)
- coherence theory (pure output states)
- resource theory of athermality

In these resource theories, the necessary and sufficient condition for correlated catalytic transformation is that $\mathfrak{D}(\rho) \geq \mathfrak{D}(\rho')$[2][3][4].

- Entanglement theory: $\mathfrak{D}$ is the relative entropy of entanglement
- Coherence theory: $\mathfrak{D}$ is the relative entropy of coherence
- Resource theory of athermality: $\mathfrak{D}$ is the non-equilibrium free energy

(In general, we do not know the necessary and sufficient conditions for $\overline{\mathcal{CCO}}$. Interesting open question!)

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Main Theorem

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We say that $\mathfrak{D}_\alpha$ is additive for the state ρ if $\mathfrak{D}_\alpha(\rho \otimes \sigma) = \mathfrak{D}_\alpha(\rho) + \mathfrak{D}_\alpha(\sigma)$ for any state σ. We define $\mathcal{Q}_\alpha(\rho) = \exp{(\alpha - 1)\mathfrak{D}_\alpha(\rho)}$.

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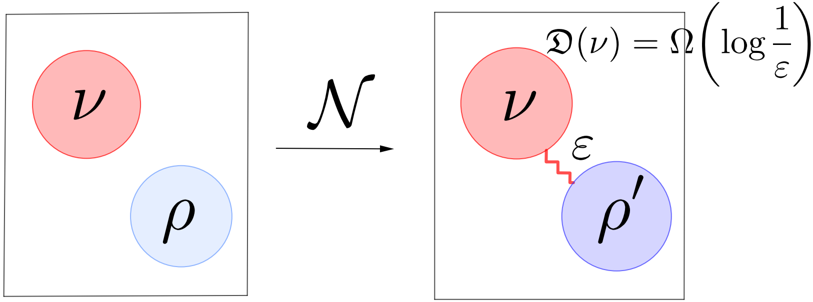
Theorem

Assume that $\rho, \rho' \in \mathcal{S}(\mathcal{H})$ and $\alpha \in [1/2, 1)$ such that $\mathfrak{D}_\alpha$ is additive for the state ρ' and $\mathfrak{D}_\alpha(\rho) < \mathfrak{D}_\alpha(\rho')$. Then, for any ε-correlated catalytic transformation with catalyst ν mapping ρ into ρ' (if $\mathfrak{D}(\rho) \geq \mathfrak{D}(\rho')$ in the resource theory we consider), we have

$$\mathfrak{D}(\nu) = \Omega\left(\log \frac{1}{\varepsilon}\right).$$

Main theorem

As the correlations vanish, the resourcefulness of the catalyst diverges.



Optimality of the bound

Our lower bound is optimal, indeed:

[5] Shiraishi, N. and Sagawa, T. (2021). Quantum Thermodynamics of Correlated-Catalytic State Conversion at Small Scale. Physical Review Letters, 126(15), 150502.

[6] van Dam, W. and Hayden, P. (2003). Universal entanglement transformations without communication. Physical Review A, 67(6), 060302.

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- $\overline{\mathcal{CCO}}$ (resource theory of athermality) [5]

$$\mathfrak{D}(\nu) = O\left(\log \frac{1}{\varepsilon}\right) \tag{6}$$

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$$\mathfrak{D}(\nu) = O\left(\log\frac{1}{\varepsilon}\right) \tag{6}$$

- Embezzlement (entanglement theory) [6]

$$\mathfrak{D}(\nu) = O\left(\log\frac{1}{\varepsilon}\right) \quad \text{and} \quad \mathfrak{D}(\nu) = \Omega\left(\log\frac{1}{\varepsilon}\right) \tag{7}$$

$\rightarrow$ we extend the lower bound to any resource theory.

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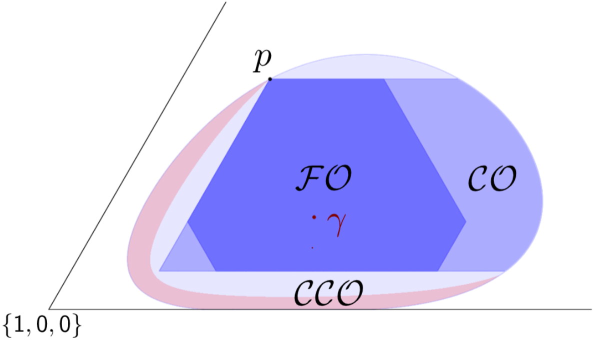
Main theorem

The condition $\mathfrak{D}_\alpha(\rho) < \mathfrak{D}_\alpha(\rho')$ for some $\alpha \in [1/2, 1)$, together with the additivity assumption, implies that the output state ρ' lies outside the set $\overline{\mathcal{CO}(\rho)}$. Hence, catalytic transformation from ρ to ρ' is possible only by allowing correlations



Check the assumptions of the main theorem

In each resource theory we find states that both satisfy $\mathfrak{D}(\rho) \geq \mathfrak{D}(\rho')$ and $\mathfrak{D}_\alpha(\rho) < \mathfrak{D}_\alpha(\rho')$ where $\alpha \in [1/2, 1)$

Is the $\mathfrak{D}_\alpha$ additive for the state ρ'?

- Resource theory of athermality: $\mathfrak{D}_\alpha$ is trivially additive for any state.
- Resource theory of coherence: $\mathfrak{D}_\alpha$ is additive for any state.[7]
- Entanglement theory: We have that whenever one state is pure, the $\mathfrak{D}_\alpha$ are additive. (Marco's talk yesterday)

Theorem

Let $\alpha \in [1/2, \infty]$, ρ be a bipartite pure state and $\mathcal{F} = SEP$. Then, for any state ν (catalyst) we have

$$\mathfrak{D}_\alpha(\rho \otimes \nu) = \mathfrak{D}_\alpha(\rho) + \mathfrak{D}_\alpha(\nu)\,. \tag{8}$$

[7] Zhu, H., Hayashi, M. and Chen, L. (2017). Coherence and entanglement measures based on Ré nyi relative entropies. Journal of Physics A: Mathematical and Theoretical, 50(47), 475303.

Proof of the main theorem - 1st ingredient: DPI for smoothed Rényi divergences

The *smoothed sandwiched quantum Rényi divergence* is defined for $\alpha \in [1/2, 1)$ as

$$\widetilde{D}_\alpha^\varepsilon(\rho\|\sigma) := \max\left\{\widetilde{D}_\alpha(\tilde{\rho}\|\sigma) : \tilde{\rho} \in \mathcal{S}_\bullet(\mathcal{H}),\ P(\tilde{\rho}, \rho) \le \varepsilon\right\},$$

where $\mathcal{S}_\bullet(\mathcal{H})$ is the set of sub-normalised states.
The use of sub-normalised states in the definition turns out to be crucial.

It is not possible to define in a smilar way a smoothed Petz Rényi divergences for $\alpha \in [0, 1)$ that satisfy the data-processing inequality.

Proof of the main theorem - 2nd ingredient: Continuity bound for the Rényi divergence

We prove the following continuity bound for the sandwiched Rényi divergences.

Proposition

Let $\alpha \in (0,1)$ *and* $\rho, \sigma \in \mathcal{S}_\bullet(\mathcal{H})$. *Then for any* $\tilde{\rho} \in \mathcal{S}_\bullet(\mathcal{H})$ *such that* $\Delta(\rho, \tilde{\rho}) \leq \varepsilon \leq \tilde{Q}_\alpha(\rho\|\sigma)^{\frac{1}{\alpha}}$ *we have*

$$|\tilde{D}_\alpha(\rho\|\sigma) - \tilde{D}_\alpha(\tilde{\rho}\|\sigma)| \leq \frac{1}{\alpha - 1} \log\left(1 - \frac{\varepsilon^\alpha}{\tilde{Q}_\alpha(\rho\|\sigma)}\right) \tag{9}$$

where we introduced generalised trace distance
$\Delta(\rho, \sigma) := \frac{1}{2}\mathrm{Tr}|\rho - \sigma| + \frac{1}{2}|\mathrm{Tr}(\rho - \sigma)|$ and the function
$\tilde{Q}_\alpha(\rho\|\sigma) = \exp{(\alpha - 1)\tilde{D}_\alpha(\rho\|\sigma)}$.

We remark that the previous bound does not depend explicitly on the dimension of the Hilbert space of the states.

Proof of the main theorem

We choose a pair of states (ρ, ρ') such that $\mathfrak{D}_\alpha(\rho) < \mathfrak{D}_\alpha(\rho')$. Therefore we have both

$$\begin{aligned} &\mathfrak{D}_\alpha(\rho \otimes \nu) < \mathfrak{D}_\alpha(\rho' \otimes \nu) \quad \text{and} \\ &\mathfrak{D}^\varepsilon_\alpha(\rho \otimes \nu) \geq \mathfrak{D}^\varepsilon_\alpha(\tau) \geq \mathfrak{D}_\alpha(\rho' \otimes \nu), \end{aligned} \tag{10}$$

However, these two inequalities lead to a tension with the continuity of $\mathfrak{D}_\alpha$, which ensures that $\mathfrak{D}_\alpha(\rho \otimes \nu)$ and $\mathfrak{D}^\varepsilon_\alpha(\rho \otimes \nu)$ are arbitrarily close as ε decreases. We then show that this tension can only be relieved if $\mathfrak{D}_\alpha(\nu)$ grows large when ε decreases.

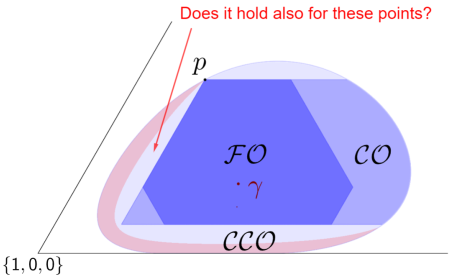
Conclusions

- We focus on the problem of catalyst preparation and our results apply to any correlated catalytic resource theory. We established that there is a trade-off between the residual correlations and the resources needed to prepare the catalyst.
- We establish that our lower bound is optimal and we extend the already existing lower bound on embezzling transformations to any resource theory.

Open questions

Open questions

- Does our result, namely that unbounded resources for the catalyst are required to achieve vanishing error, hold for any state in $\overline{\mathcal{CCO}} \setminus \overline{\mathcal{CO}}$?



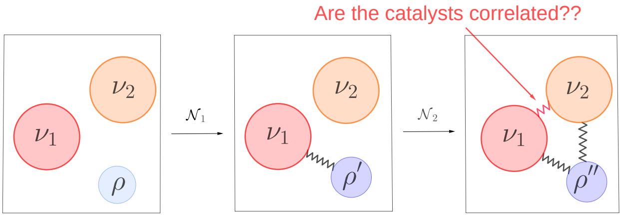
A characterization of the sets $\overline{\mathcal{CO}}$ and $\overline{\mathcal{CCO}}$, and therefore of the set $\overline{\mathcal{CCO}} \setminus \overline{\mathcal{CO}}$, is not known for many resource theories. Hence, the range of applicability of our main theorem and whether unbounded resources for the catalyst are required in such theories are still open questions.

Open questions

- Composability of correlated catalytic transformations

$$\rho \xrightarrow{\nu_1} \rho' \xrightarrow{\nu_2} \rho'' \tag{11}$$

Are ν_1 and ν_2 now correlated? Are correlated catalytic transformations closed under composition?



Thanks for your attention!