Improved Classical Simulation of Quantum Circuits Dominated by Fermionic Linear Optical Gates

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December 8, 2022

Foundation for Polish Science

(Pre)motivation

Foundational

What (if anything) separates quantum computational power from classical?

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Foundational

What (if anything) separates quantum computational power from classical?

"Practical"

What is the best way to classically simulate a given quantum circuit?

Introduction

What is a quantum computer?

For the purposes of this talk:

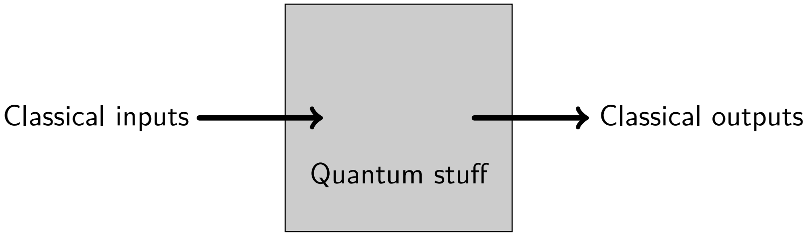

Figure 1: A quantum computer

Computational resource theories

What are they

- A set S of quantum states
- A set O of operations which preserve S
- A set M of observables
- A set of questions you can ask, and get answers to efficiently (polynomial time)

Examples

- Clifford/stabilizer circuits
- FLO/matchgates circuits
- Incoherent circuits
- Log-space universal quantum circuits
- Tensor-product circuits

A landsape of (quantum) computation

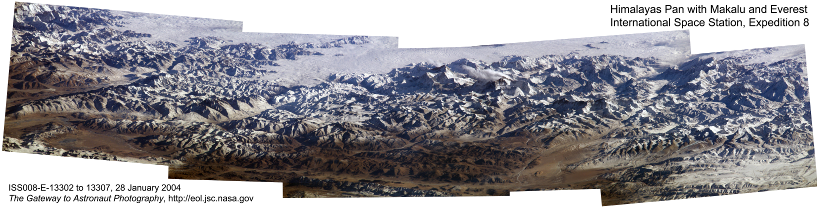

Figure 2: Mosaic image of the Himalayas taken by astronauts on ISS. Image credit: NASA. The landsape represents the space of possible quantum computations with the height of a point representing the difficulty or simulation cost.

Basecamps

(a) Base camp on the Nepalese side of Sagarmāthā. CC licence, Author: Peellden.

(b) Base camp on the Tibetan side of Chomolungma. CC licence, Author: Gunther Hagleitner.

Figure 3: Two base camps in the vicinity of the mountain known as ཇོ་མོ་གླང་མ (Chomolungma), सगरमाथा (Sagarmāthā), 珠穆朗玛峰 (Zhūmùlǎngmǎ Fēng) and Everest. Basecamps are relatively easy to get to and make climbing nearby mountains easier. Computational resource theories are easy to simulate and make simulating "nearby" computations easier.

Aim

Aim: Estimate $p := |\langle y|U|x\rangle|^2$

Where

1. x and y are computational basis vectors
2. U consists of
   - many Fermionic linear optical (FLO) unitaries
   - some non-FLO Controlled-phase gates

Simulating non-free states

Density matrix

Write non-free state as linear (not convex) combination of free states

$$\rho = \sum_i q_i \sigma_i. \tag{1}$$

Renormalize to get a probability distribution

$$\rho = ||q||_1 \sum_i \frac{|q_i|}{||q||_1} (-1)^{s_i} \sigma_i. \tag{2}$$

Statevector

Write non-free state as superpositions of free states

$$|\psi\rangle = \sum_i \alpha_i |f_i\rangle. \tag{3}$$

Renormalize to get a probability distribution

$$|\psi\rangle = ||\alpha||_1 \sum_i \frac{|\alpha_i|}{||\alpha||_1} e^{i\theta_i} |f_i\rangle \tag{4}$$

Two different 1-norms

Simulation cost

- $||q||_1$ is called *negativity*
- $||\alpha||_1^2$ is called *extent*
- (Informally) you can either get simulation runtime scaling in extent or negativity squared
- In the examples I know $||\alpha||_1^2 \leq ||q||_1^2$.

Extent (general)

Given some resource theory, with set of free states S

$$\xi_S(\psi) = \inf\left\{ ||\alpha||_1^2 \,\middle|\, \psi = \sum_{s\in S} \alpha_s s \right\} \tag{5}$$

Our contribution

A simulation algorithm for universal quantum circuits, with runtime

1. Linear in the Fermionic linear optical extent
2. Polynomial in all other relevant parameters

Fermionic linear optics (FLO)

Fermionic linear optics overview (1)

Creation and annihilation operators

Define Fermionic creation ($a_i^\dagger$) and annihilation (a_i) operators for $i = 0 \ldots n-1$ with

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \qquad \{a_i, a_j^\dagger\} = \delta_{ij}, \tag{6}$$

Majorana Fermion operators

Now define the *Majorana Fermion* operators

$$c_{2i} = a_i + a_i^\dagger \tag{7}$$

$$c_{2i+1} = -i(a_i - a_i^\dagger), \tag{8}$$

Note

$$\{c_i, c_j\} = 2\delta_{i,j} \tag{9}$$

Explicit Majorana Fermion operators (J-W version)

$$c_0 = X \otimes I \dots \qquad c_1 = Y \otimes I \dots \tag{10}$$

$$c_2 = Z \otimes X \otimes I \dots \qquad c_3 = Z \otimes Y \otimes I \dots \tag{11}$$

$$c_4 = Z \otimes Z \otimes X \otimes I \dots \qquad c_5 = Z \otimes Z \otimes Y \otimes I \dots \tag{12}$$

$$\vdots$$

$$c_2 c_8 = (Z \otimes X \otimes I \dots)(Z \otimes Z \otimes Z \otimes Z \otimes X \otimes I \dots) \tag{13}$$

$$= -iI \otimes Y \otimes Z \otimes Z \otimes X \otimes I \dots \tag{14}$$

Fermionic linear optics overview (2)

A unitary U is *Fermionic* if (and only if)

$$U c_i U^\dagger = \sum_j R_{ij} c_j, \tag{15}$$

for a real, special orthogonal, $2n \times 2n$ matrix R.

Terhal and DiVincenzo [1]

Theorem

Given a Fermionic linear optic circuit U and computational basis states $|x\rangle$, $|y\rangle$ there is a polynomial-time classical algorithm which computes

$$p = |\langle y|U|x\rangle|^2 . \tag{16}$$

[1] Classical simulation of noninteracting-fermion quantum circuits, Barbara M. Terhal and David P. DiVincenzo, Phys. Rev. A **65** 032325 (2002)

Magic states for FLO

Theorem

FLO circuits supplemented by single qubit measurements and magic states of the form[2][3]

$$|M_\phi\rangle = \frac{1}{2}\left(|0000\rangle + |1100\rangle + |0011\rangle + e^{i\phi}|1111\rangle\right), \qquad (17)$$

are universal for quantum computation.

[2] All Pure Fermionic Non-Gaussian States Are Magic States for Matchgate Computations, M. Hebenstreit, R. Jozsa, B. Kraus, S. Strelchuk, and M. Yoganathan, Phys. Rev. Lett. 123, 080503 (2019)

[3] Universal quantum computation with the $\nu = \frac{5}{2}$ fractional quantum Hall state Sergey Bravyi Phys. Rev. A 73, 042313 (2006)

Decomposition

Lemma

The magic state $|M_\phi\rangle$ admits the decomposition

$$|M_\phi\rangle = \cos\left(\frac{\phi}{4}\right)\boxed{\frac{1}{2}\left(e^{-\frac{i\phi}{4}}|0000\rangle + e^{\frac{i\phi}{4}}|1100\rangle + e^{\frac{i\phi}{4}}|0011\rangle + e^{\frac{3i\phi}{4}}|1111\rangle\right)}$$
$$+ i\sin\left(\frac{\phi}{4}\right)\boxed{\frac{1}{2}\left(e^{\frac{-i\phi}{4}}|0000\rangle - e^{\frac{i\phi}{4}}|1100\rangle - e^{\frac{i\phi}{4}}|0011\rangle + e^{\frac{3i\phi}{4}}|1111\rangle\right)}.$$

The two highlighted terms are each Fermionic Gaussian states (states obtainable by the action of matchgate circuits on an initial $|0\rangle$ vacuum state).

Phase-sensitive FLO simulation

Phase sensitive Fermionic linear optics simulator

Basic issue

Efficient simulation is (partly) based on "adjoint action"[4]

$$U c_i U^\dagger = \sum_j R_{ij} c_j. \tag{18}$$

Any phase information is lost in this representation.

[4] However, see Complexity of Quantum Impurity Problems, Sergey Bravyi and David Gosset, Communications in Mathematical Physics **356** 451–500 (2017).

Phase sensitive FLO simulator

Passive-active (PA) form

An arbitrary FLO state $|\psi\rangle$ may be expressed in the form

$$|\psi\rangle = \omega K \prod_j \left(\cos(2\lambda_j)\, I + \sin(2\lambda_j)\, a_{2j}^\dagger a_{2j+1}^\dagger\right)|0\rangle. \tag{19}$$

Where,

1. $\omega \in \mathbb{C}$
2. K is a passive FLO unitary (commutes with number operator)
3. $\lambda_j \in \mathbb{R}$
4. $a_{2j}^\dagger a_{2j+1}^\dagger$ makes two Fermions on adjacent sites

Operations

We can

1. Apply FLO unitaries
2. Apply computational-basis projectors
3. Compute computational-basis inner products

all in polynomial time on a classical computer.

Universal simulation algorithm

Algorithm (1)

Input

1. Computational basis vectors x and y
2. Quantum circuit U consisting of k controlled phase gates & $\mathrm{poly}(k)$ FLO gates
3. $\epsilon, \delta > 0$, the error and failure rate you are willing to tolerate.

Aim: Estimate $p := |\langle y|U|x\rangle|^2$ to precision ϵ with probability greater than $1 - \delta$.

How does the algorithm work?

Key ideas

1. Gadgetization - replace non-FLO gates with FLO gates and input magic states
2. Decomposition - write each magic state as a superposition of FLO states
3. Sampling - use concentration inequality to bound how many FLO states to sample

What controls the runtime?

Hoeffding's Inequality

Single magic state:

$$|M_\phi\rangle = \frac{1}{2}\left(|0000\rangle + |1100\rangle + |0011\rangle + e^{i\phi}|1111\rangle\right) \tag{20}$$

$$= \cos\left(\frac{\phi}{4}\right)|A(\phi)\rangle + i\sin\left(\frac{\phi}{4}\right)|B(\phi)\rangle \tag{21}$$

$$= \xi(\phi)^{\frac{1}{2}}\left(P_\phi(A)|A(\phi)\rangle + P_\phi(B)i|B(\phi)\rangle\right), \tag{22}$$

where $\xi(\phi) := \xi(|M_\phi\rangle)$

Multiple magic states: $\xi^* = \prod_j \xi(\phi_j)$

Algorithm (2)

Output

An estimate $\hat{p}$ satisfying

$$\Pr(|\hat{p} - p| \geq \epsilon) \leq \delta. \tag{23}$$

Runtime: $O\left(\frac{\xi^*}{\epsilon^2} \log\left(\frac{1}{\delta}\right) \mathrm{poly}(n, k)\right)$

Comparison to prior work

Prior work

Shigeo Hakkaku, Yuichiro Tashima, Kosuke Mitarai, Wataru Mizukami, and Keisuke Fujii. "Quantifying Fermionic Nonlinearity of Quantum Circuits". Dec. 2021. `https://arxiv.org/abs/2111.14599`

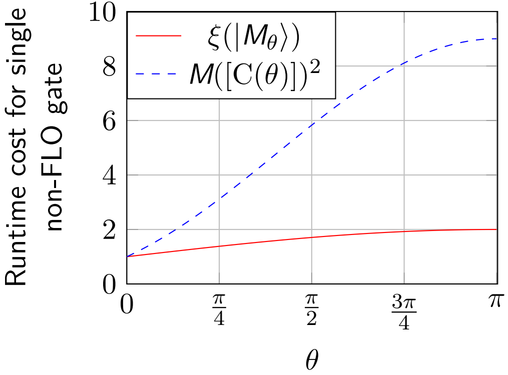

Figure 4: A comparison of the runtime cost of adding non-FLO controlled phase to a quantum circuit for our simulation method, and the algorithm of Ref.[6]

Outlook

Open questions & future work

FLO

- The FLO-extent is submultiplicative, (when) is it multiplicative?
- Links between FLO and Clifford/stabilizer subtheory
- Improvements to estimation algorithm
- Adaption to sampling
- Applications

Broader questions

- What are the relations between different classically simulable subtheories?
- Is this the “right” way to go about classical simulation of quantum mechanics?
- Are there other interesting efficiently simulable subtheories we don’t know about?

Thank you for your time