Universality of Local Free Operations in Resource Theories

When can free operations be realized with 2-local free operations?

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It is often claimed that a resource theory characterizes a physical property of quantum systems as a "resource". To justify this we should ask

- Is there any fundamental or practical restrictions on the set of realizable operations selecting this particular set of free operations?
- How can we implement free operations? Is it really possible to implement them without consuming the resource under consideration?

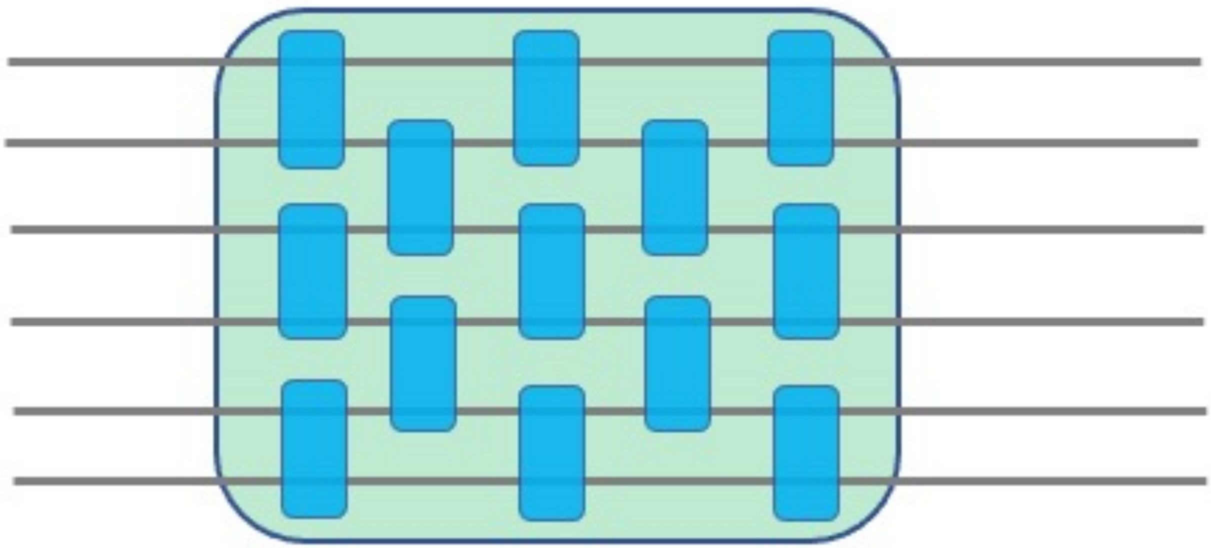
The "golden rule" of resource theories is not sufficient to answer these questions.

Criteria for physically realizable free operations

- **Existence of Free Dilation:** A free operation should be realizable via free unitary transformations, ancilla systems in free states, and, possibly, free projective measurements.

Chitambar-Gour 2016, Marvian-Spekkens 2016

- **Universality of Local Free Unitaries**



Universality: Any unitary transformation on a composite system can be realized *exactly* by a finite sequence of 2-local unitaries on the subsystems.

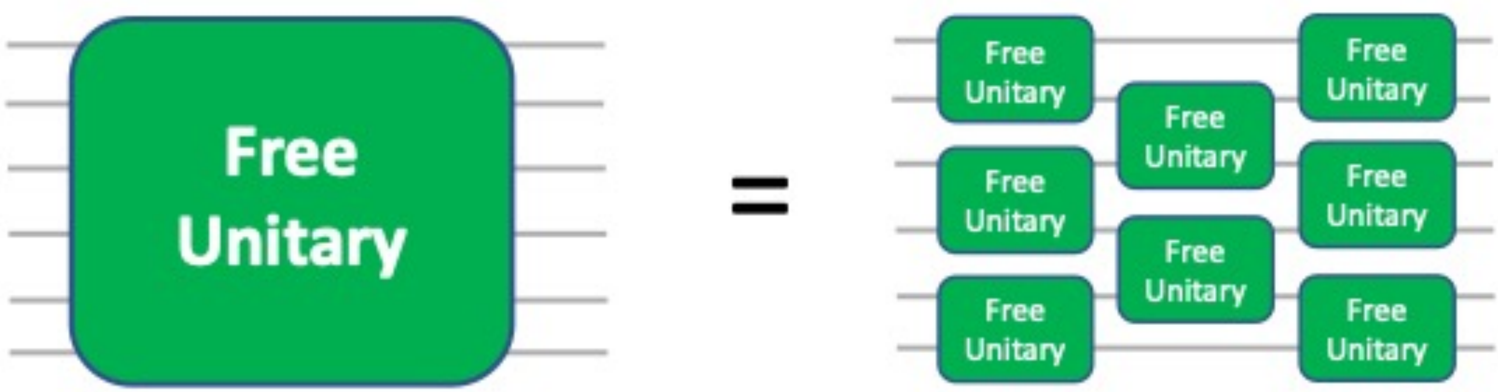
Universality with 2-qubits gates (DiVincenzo 1995), Almost all 2-qubits gates are universal (Lloyd 1995, Deutsch-Barenco-Ekert 1995)

Criterion: Universality of local free unitaries

Free global unitaries should be realizable with 2-local free unitaries, or, more generally, k-local free unitaries with a fixed k independent of the number of subsystems.

From fundamental and practical points of view, we are restricted to interactions that couple a few subsystems together.

If this universality does not hold, then implementing some free global unitaries may require consuming the resource under consideration.



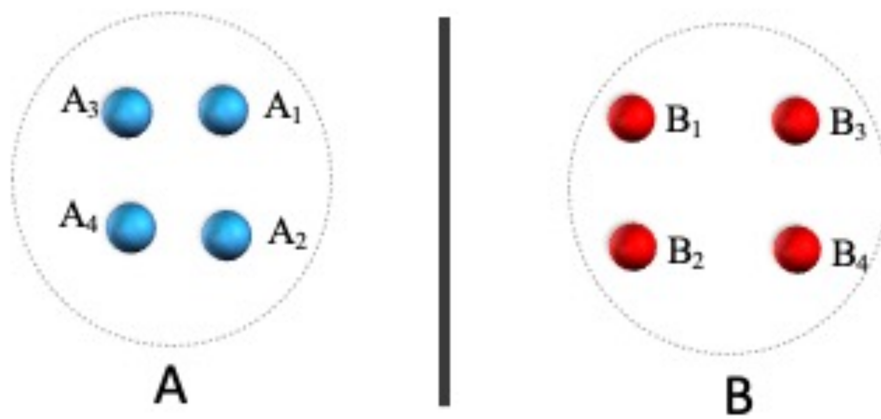
Resource theories with universal local free operations

Example: Resource Theory of Magic States

The Clifford group is generated by Hadamard, Phase and CNOT gates.
Therefore, 2-local free unitaries are universal.

Example: Resource Theory of Entanglement

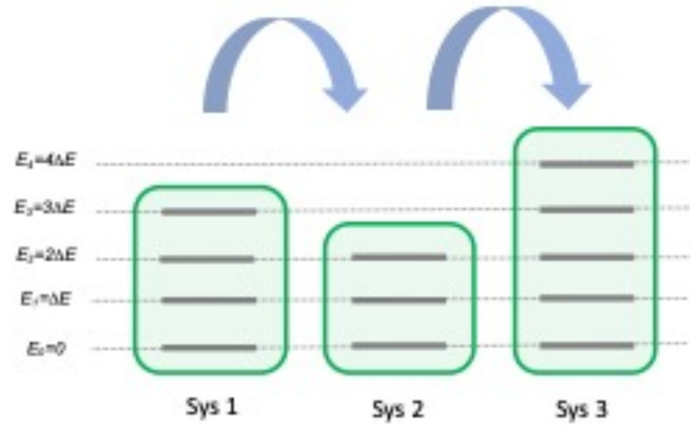
2-local free unitaries are universal.



How about Quantum Thermodynamics?

V is **energy-conserving** if $[V, H_{\text{tot}}] = 0$

In the resource theory of Quantum Thermodynamics it is assumed that any energy-conserving unitary can be implemented with **negligible thermodynamic cost**.



$$H_{\text{tot}} = H_1 + H_2 + H_3$$

How can we implement a general energy-conserving unitary?

Can we obtain all energy-conserving unitaries by combining 2-local energy-conserving unitaries?

Energy can be redistributed by a sequence of **2-local energy-conserving** operations.

Is there any **hidden thermodynamic cost** for implementing a general energy-conserving unitary, using local energy-conserving unitaries?

Part I: Symmetric Unitaries

- **Energy-conserving unitaries for periodic systems, U(1)-invariant unitaries**
  Resource theories of quantum thermodynamics, U(1) asymmetry, unspeakable coherence

- **Rotationally-invariant unitaries on qubits**
  Resource theories of asymmetry for SU(2) symmetry with qubits and quantum thermodynamics with SU(2)-conserved charges

- **SU(*d*)-invariant unitaries on qudits**
  Resource theory of asymmetry for SU(d) symmetry with qudits qubits (quantum thermodynamics with SU(d)-conserved charges)

Part II

- **Incoherent Unitaries**
  Resource theories of speakable coherence

- **Diagonal Unitaries, Energy-conserving unitaries for systems with non-degenerate Hamiltonians**
  Resource theory of quantum thermodynamics for non-degenerate Hamiltonians

Example: qubits with U(1) symmetry

For a system with *n* qubits consider the U(1) group corresponding to rotations around z axis

Symmetric unitaries $V(e^{i\theta Z})^{\otimes n} = (e^{i\theta Z})^{\otimes n} V \qquad \theta \in [0, 2\pi)$

Conservation of total L_z (Hamming weight) $V(\sum_{j=1}^{n} Z_j)V^\dagger = \sum_{j=1}^{n} Z_j$

Symmetric unitaries form a group.

n=1 $\begin{pmatrix} e^{i\phi_0} & \\ & e^{i\phi_1} \end{pmatrix} \begin{matrix} |0\rangle \\ |1\rangle \end{matrix}$

n=2 $\begin{pmatrix} e^{i\phi_0} & & \\ & V_1 & \\ & & e^{i\phi_2} \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$

n=3 $\begin{pmatrix} e^{i\phi_0} & & & \\ & V_1 & & \\ & & V_2 & \\ & & & e^{i\phi_3} \end{pmatrix} \begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |100\rangle \\ |011\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix}$

Do 2-local symmetric unitaries generate all symmetric unitaries?

General Symmetry

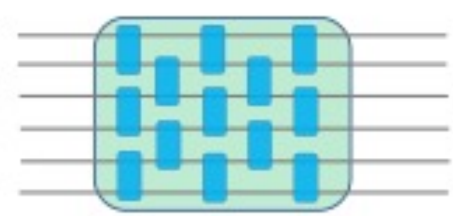
For a general group *G*, let $\{u_j(g) : g \in G\}$ be the representation of symmetry on site *j*.

We say a unitary *V* is **symmetric** with respect to the symmetry described by group *G*, if

$$\forall g \in G: \quad V \bigotimes_{j=1}^{n} u_j(g) = \bigotimes_{j=1}^{n} u_j(g) V$$

Or, equivalently

$$\forall g \in G: \quad [\bigotimes_{j=1}^{n} u_j(g)] V [\bigotimes_{j=1}^{n} u_j^\dagger(g)] = V$$



A no-go theorem: Non-universality of local symmetric unitaries

- In the case of continuous symmetries, generic symmetric unitaries cannot be implemented using local symmetric unitaries on the subsystems.

  In fact, in the case of U(1) symmetry, k-local symmetric unitaries do not generate all (k+1)-local symmetric unitaries.

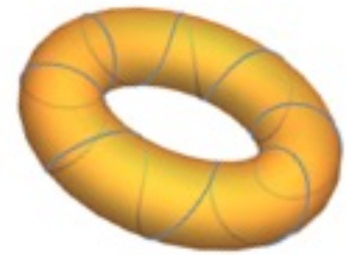
Torus: Symmetric unitaries
Helix: The subgroup generated by k-local symmetric unitaries

- The group of symmetric unitaries and its subgroup generated by k-local symmetric unitaries are both compact connected Lie groups, and hence closed manifolds.

- The gap between the dimensions of the two manifolds increases with the system size.

IM, *Restrictions on realizable unitary operations imposed by symmetry and locality*, Nature Physics (2022).

Example: U(1) Symmetry



The group generated by k-local U(1)-invariant unitaries

$$\dim(\mathcal{V}_n^{\mathrm{U}(1)}) - \dim(\mathcal{V}_k^{\mathrm{U}(1)}) = n - k$$

Example: *n*=3 qubits

Group of all U(1)-invariant unitaries: **20 D**

Subgroup generated by 2-local U(1)-invariant unitaries: **19 D**

The family of unitaries $\{e^{-i\theta Z^{\otimes 3}} : \theta \in [0, 2\pi)\}$ cannot be generated using 2-local U(1)-invariant unitaries (except a finite set of unitaries).

IM, *Restrictions on realizable unitary operations imposed by symmetry and locality*, Nature Physics (2022).

Example: SU(2) Symmetry

The group generated by k-local SU(2)-invariant unitaries

$$\dim(\mathcal{V}_n^{\mathrm{SU}(2)}) - \dim(\mathcal{V}_k^{\mathrm{SU}(2)}) = \lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor$$

Example: On a system with 4 qubits, the family of unitaries generated by rotationally-invariant Hamiltonian *H* can be realized using the exchange interaction, up to a global phase, if and only if

$$15\,\mathrm{Tr}(H\Pi_{j=0}) - 5\,\mathrm{Tr}(H\Pi_{j=1}) + 3\,\mathrm{Tr}(H\Pi_{j=2}) = 0$$

Π_j is the projector to the subspace with angular momentum *j*

	Angular Momentum					
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$l=0$ body	1	1	1	1	1	1
$l=2$ body	-15	-11	-3	9	25	45
$l=4$ body	150	70	-42	-90	70	630
$l=6$ body	-1050	-210	462	-90	-1050	3150
$l=8$ body	4725	-315	-1323	2565	-3675	4725
$l=10$ body	-10395	3465	-2079	1485	-1155	945

n=5 qubits

$$c_l(j) = \frac{1}{2^{l/2}(n-l)!} \sum_{r=0}^{l/2} \frac{(-4)^r r!(n-2r)!}{(l/2-r)!} \binom{\frac{n}{2}+j}{r} \binom{\frac{n}{2}-j}{r} \frac{\frac{n}{2}+j+1}{\frac{n}{2}+j+1-r}$$

IM, H. Liu, and A. Hulse, *Rotationally-Invariant Circuits: Universality with the exchange interaction and two ancilla qubits*, arXiv:2202.01963 (2022).

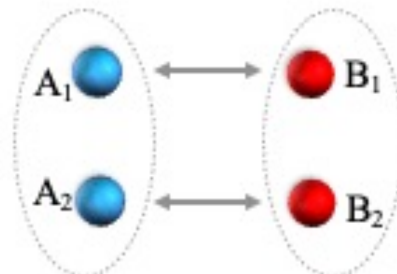
These constraints forbid certain unitaries that have been previously used in the resource theories of quantum thermodynamics and asymmetry.

Example:

Multi-qubit swap Hamiltonian cannot be realized with 2-local rotationally-invariant unitaries.

For example, 4-qubit swap Hamiltonian does not satisfy

$$15\,\mathrm{Tr}(H\Pi_{j=0}) - 5\,\mathrm{Tr}(H\Pi_{j=1}) + 3\,\mathrm{Tr}(H\Pi_{j=2}) = 0$$



$$S_{AB} = S_{A_1B_1} \otimes S_{A_2B_2}$$

Swap Hamiltonian

IM, H. Liu, and A. Hulse, *Rotationally-Invariant Circuits: Universality with the exchange interaction and two ancilla qubits*, arXiv:2202.01963 (2022).

General groups

Grows unboundedly for continuous symmetries

$$\dim(\mathcal{V}_n^G) - \dim(\mathcal{V}_k^G) \geq |\mathrm{Irreps}_G(n)| - |\mathrm{Irreps}_G(k)|$$

$\mathcal{V}_n^G$: The Lie group of all symmetric unitaries $\{V : VV^\dagger = I, [V, u(g)^{\otimes n}] = 0 \ : \forall g \in G\}$

$\mathcal{V}_k^G$: The subgroup generated by *k*-local symmetric unitaries

$\dim(\mathcal{V}_k^G)$: Dimension of the corresponding manifold

$|\mathrm{Irreps}_G(k)|$: Number of inequivalent irreducible representations of *G* appearing in $\{u(g)^{\otimes k} : g \in G\}$ the representation of symmetry on *k* subsystems.

For the example of *n* qubits with U(1) symmetry: $|\mathrm{Irreps}^{U(1)}(k)| = k + 1$

IM, *Restrictions on realizable unitary operations imposed by symmetry and locality*, Nature Physics (2022).

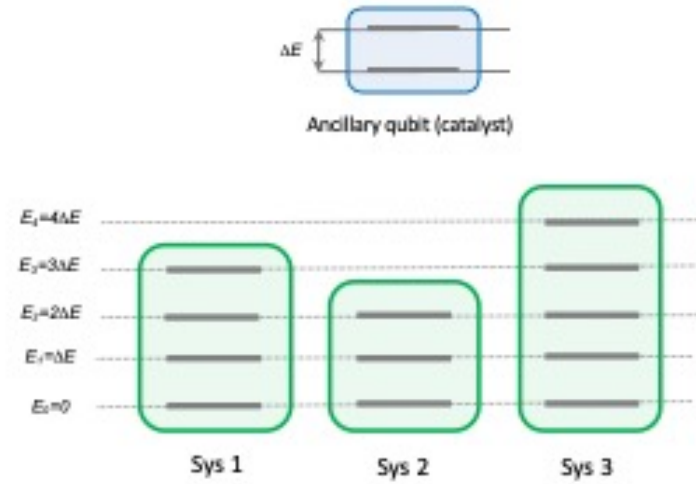
Circumventing the no-go theorem with ancilla qubits

U(1) and SU(2) symmetries

Quantum Thermodynamics with Local Energy-Conserving Unitaries

No-go theorem: Generic global energy-conserving unitaries cannot be implemented using local energy-conserving unitaries.

Theorem: Consider a finite set of closed systems with the property that for each system the gap between any consecutive pairs of energy levels is ΔE . Then, any global energy-conserving unitary transformation on these systems can be implemented by a finite sequence of 2-local energy-conserving unitaries, provided that one can use a single ancillary qubit (catalyst) with energy gap ΔE between its two levels.

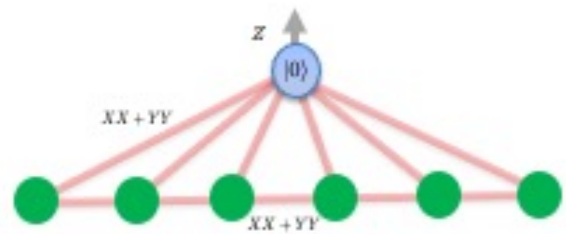


IM, *Restrictions on realizable unitary operations imposed by symmetry and locality*, Nature Physics (2022).

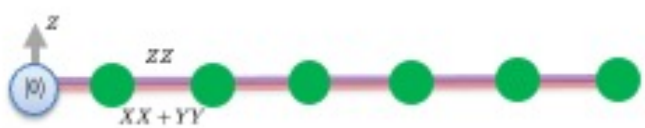
General U(1)-invariant unitary with Hamiltonians *XX+YY* and local *Z*

Using 2-local Hamiltonian *X X + Y Y* and single-qubit Pauli *Z*, which are both invariant under rotations around z, it is possible to implement all unitaries which are invariant under this symmetry, provided that one can employ an ancillary qubit.

XX+YY and *Z*



XX+YY , *ZZ* and *Z*



IM, *Restrictions on realizable unitary operations imposed by symmetry and locality*, Nature Physics (2022).

Circumventing the no-go theorem for SU(2) symmetry

Theorem: Any rotationally-invariant unitary V on qubits can be implemented using the Heisenberg exchange interaction *XX*+*YY*+*ZZ* and 2 ancilla qubits, i.e.,

$$\widetilde{V}(|\psi\rangle \otimes |00\rangle_{ab}) = (V|\psi\rangle) \otimes |00\rangle_{ab}$$

where unitary $\tilde{V}$ can be implemented using the Heisenberg exchange interaction.

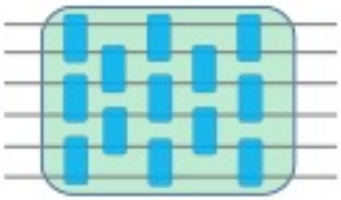
Furthermore, a single ancilla qubit is not sufficient to remove the constraints imposed by locality.

Remark: The state of ancilla qubits does not need to be symmetry-breaking. It can be any density operator with support restricted to the triplet subspace. Also, a variant of this scheme works with ancilla qubits in the singlet state.

IM, H. Liu, and A. Hulse, *Rotationally-Invariant Circuits: Universality with the exchange interaction and two ancilla qubits*, arXiv:2202.01963 (2022).

Qudits with SU(*d*) symmetry

Total Hilbert space of *n* qudits $(\mathbb{C}^d)^{\otimes n}$



$$[V, U^{\otimes n}] = 0, \ : U \in \mathrm{SU}(d)$$

A surprising distinction between d=2 and d>2

***d*=2:** Locality only restricts the relative phases between different conserved sectors.

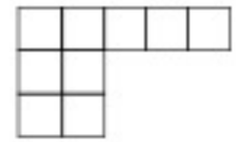
$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_\lambda \mathbb{C}^{d_\lambda} \otimes \mathbb{C}^{m(n,\lambda)}$$

***d*>2**: In addition to similar constraints on relative phases between different conserved sectors, locality also restricts the generated unitaries inside these conserved subspaces.

IM, H. Liu, and A. Hulse, *Qudit circuits with SU(d) symmetry: Locality imposes additional conservation laws*, arXiv:2105.12877 (2021).

Example: 6 Qutrits with SU(3) symmetry

Total Hilbert space: $(\mathbb{C}^3)^{\otimes 6}$



$$|\text{singlet}\rangle = (|0\rangle \wedge |1\rangle \wedge |2\rangle) = \frac{1}{\sqrt{6}} \sum_{ijk=0}^{2} \epsilon_{ijk} |i\rangle|j\rangle|k\rangle \in (\mathbb{C}^3)^{\otimes 3}$$

The joint state is restricted to a subspace with a single irrep of SU(3).

$$|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} \xrightarrow[\text{SU(3)-invariant unitaries}]{} \alpha|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} + \beta|0\rangle^{\otimes 3} \otimes |\text{singlet}\rangle$$

$$|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} \xrightarrow[\textbf{2-local}\text{ SU(3)-invariant unitaries}]{} \begin{cases} |\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} \\ |0\rangle^{\otimes 3} \otimes |\text{singlet}\rangle \end{cases}$$

$$\underbrace{|\text{singlet}\rangle}_{\text{Catalyst}} \otimes \left[|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3}\right] \xrightarrow[\textbf{2-local}\text{ SU(3)-invariant unitaries}]{} \left[\alpha|\text{singlet}\rangle \otimes |0\rangle^{\otimes 3} + \beta|0\rangle^{\otimes 3} \otimes |\text{singlet}\rangle\right] \otimes \underbrace{|\text{singlet}\rangle}_{\text{Catalyst}}$$

Circumventing the no-go theorem for SU(d) symmetry

Theorem: Any SU(d)-invariant unitary on qudit systems can be realized with 2-local SU(d)-invariant unitaries, provided that one can use $3d$ ancilla qudits.

A. Hulse, Hanqing Liu, IM, Circuits with SU(d) symmetry, under preparation

IM, H. Liu, and A. Hulse, *Qudit circuits with SU(d) symmetry: Locality imposes additional conservation laws*, arXiv:2105.12877 (2021).

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- **U(1)-invariant unitaries, Energy-conserving unitaries for periodic systems**
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Part II

- **Incoherent Unitaries**
  Resource theories of speakable coherence

- **Diagonal Unitaries**
  Resource theory of quantum thermodynamics for non-degenerate Hamiltonians

Incoherent Unitaries

$$U = \sum_j e^{i\phi_j} |\pi(j)\rangle\langle j|$$

Incoherent unitaries

A general incoherent unitary **cannot be realized with 2-local** incoherent unitaries, even if one is allowed to use ancilla qubits and catalysts.

On the other hand, a general incoherent unitary **can be realized with 3-local** incoherent unitaries and 2 ancillary qubits.

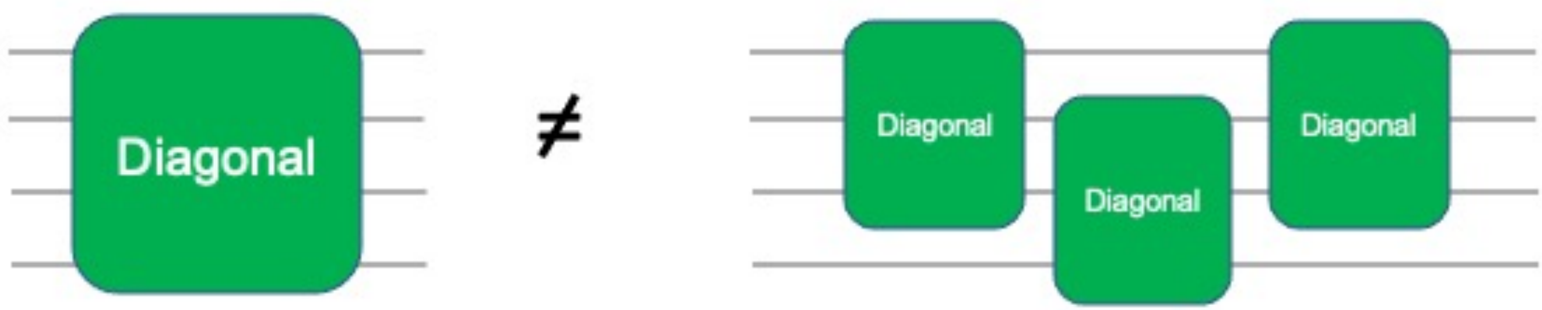
Diagonal Unitaries

Diagonal unitaries

$$U = \sum_j e^{i\phi(j)} |j\rangle\langle j|$$

They appear, e.g., in Quantum Thermodynamics for systems with non-degenerate Hamiltonians.

A general diagonal unitary on *n* systems **cannot be realized with (*n*-1)-local diagonal unitaries**, even if one is allowed to use arbitrary number of ancilla qubits and catalysts.

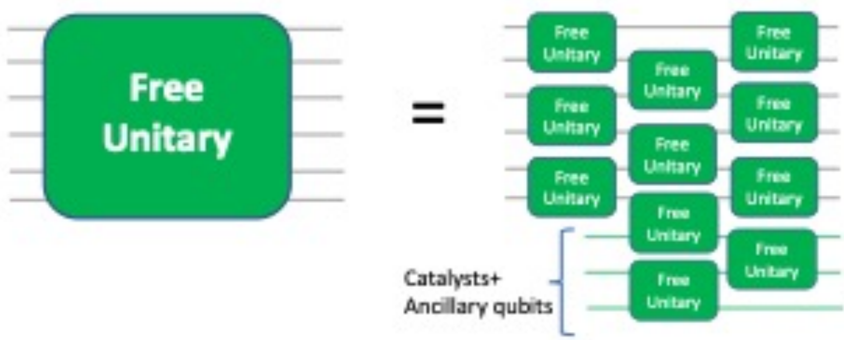


Summary

Family of unitaries	Universal	Not Universal
U(1)-invariant unitaries	2-local unitaries + 1 ancilla qubit	2-local unitaries
Incoherent unitaries	3-local unitaries + 2 ancilla qubits	2-local unitaries + arbitrary ancilla qubits 3-local unitaries + 1 ancilla qubit
Diagonal unitaries	*n*-local unitaries on *n* qudits	(*n*-1)-local unitaries on *n* qudits + ancilla qudits

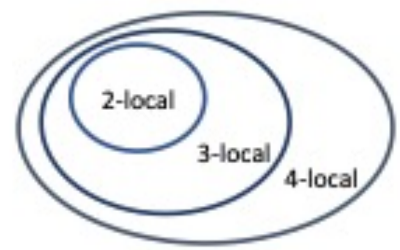
Universality Criterion (v2)

Free global unitaries should be realizable with k-local free unitaries with a fixed k, together with ancillary qubits and catalysts (and, possibly local free projective measurements).



Future Directions

- Hierarchy of resources theories with *k*-local unitaries
- Complexity of free operations



Thank you for your attention.

1) IM, *Restrictions on realizable unitary operations imposed by symmetry and locality*, Nature Physics (2022).
2) IM, H. Liu, and A. Hulse, *Rotationally-Invariant Circuits: Universality with the exchange interaction and two ancilla qubits*, arXiv:2202.01963 (2022).
3) IM, H. Liu, and A. Hulse, *Qudit circuits with SU(d) symmetry: Locality imposes additional conservation laws*, arXiv:2105.12877 (2021).

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PhD and Postdoc positions available at Duke Quantum Center