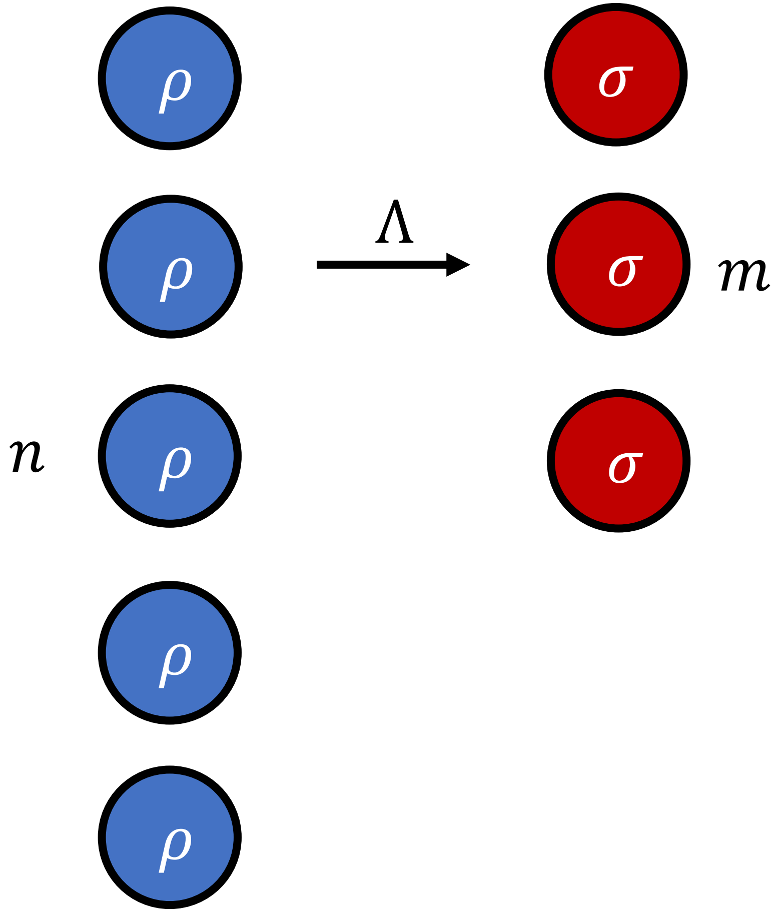
Finite-size catalysis in quantum resource theories

Patryk Lipka-Bartosik and Kamil Korzekwa

Soon on arXiv!

Collective transformations



$$\Lambda(\rho^{\otimes n}) \approx \sigma^{\otimes m}$$

Basic question: What is the optimal rate?

$$R_\epsilon^n(\rho,\sigma) = \sup\left\{ \frac{m}{n} \,\middle|\, \inf_{\Lambda\in F} \left\|\Lambda(\rho^{\otimes n}) - \sigma^{\otimes m}\right\| < \epsilon \right\}$$

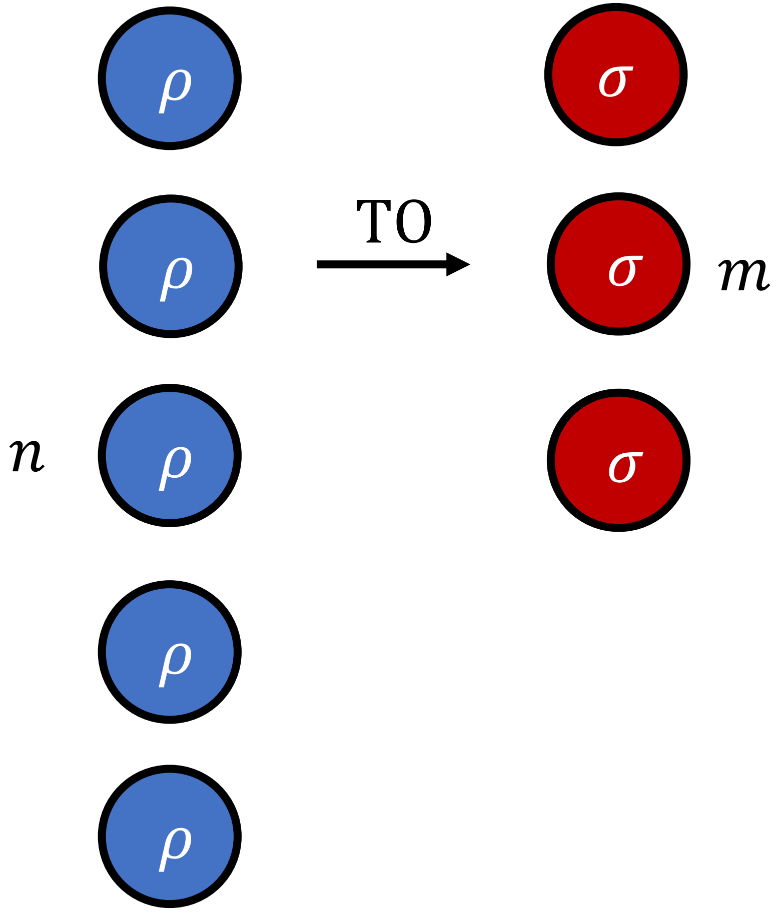
$$R_\epsilon^\infty(\rho,\sigma) \coloneqq \lim_{n\to\infty} R_\epsilon^n(\rho,\sigma)$$

M. Horodecki, J. Oppenheim, **Int. J. Mod. Phys. B** (2013)

G. Gour et. al., **Phys. Rep.** (2015)

E. Chitambar, G. Gour, **Rev. Mod. Phys.** (2019)

Collective transformations



$$\mathrm{TO}\left(\rho^{\otimes n}\right) \approx \sigma^{\otimes m}$$

Basic question: What is the optimal rate?

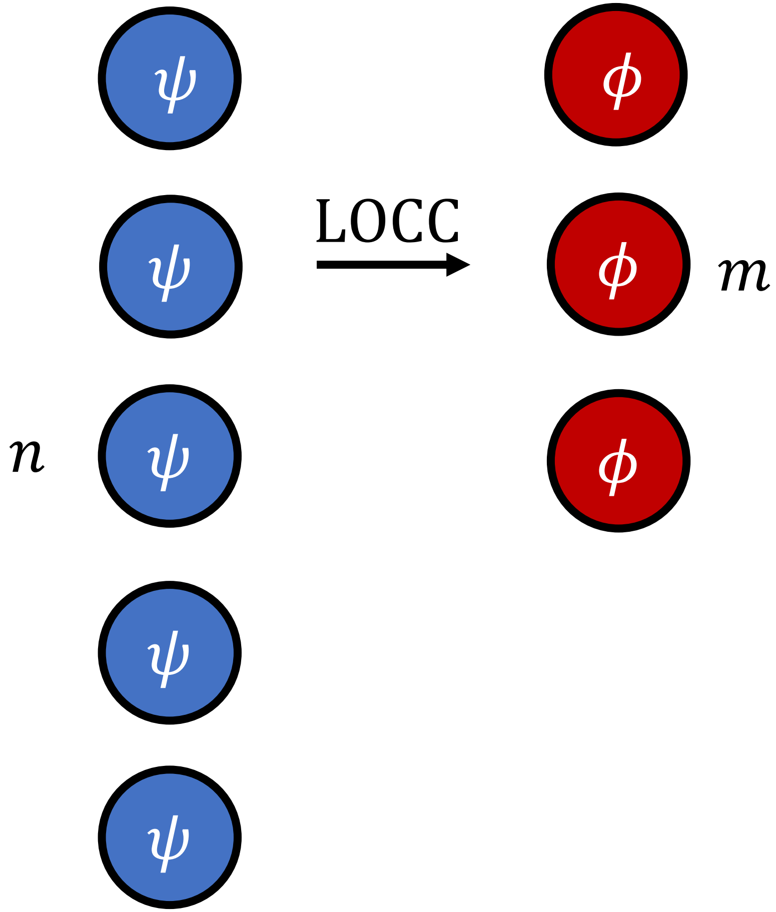
Example: Thermodynamics (thermal operations)

$$R_\epsilon^\infty(\rho,\sigma) := \lim_{n\to\infty} R_\epsilon^n(\rho,\sigma) \qquad \gamma \propto e^{-\beta H}$$

$$= \frac{D(\rho||\gamma)}{D(\sigma||\gamma)}$$

F. Brandao et al., **Phys. Rev. Lett.**, 250404 (2013)

Collective transformations



$$\mathrm{LOCC}\left(\psi_{AB}^{\otimes n}\right) \approx \phi_{AB}^{\otimes m}$$

Basic question: What is the optimal rate?

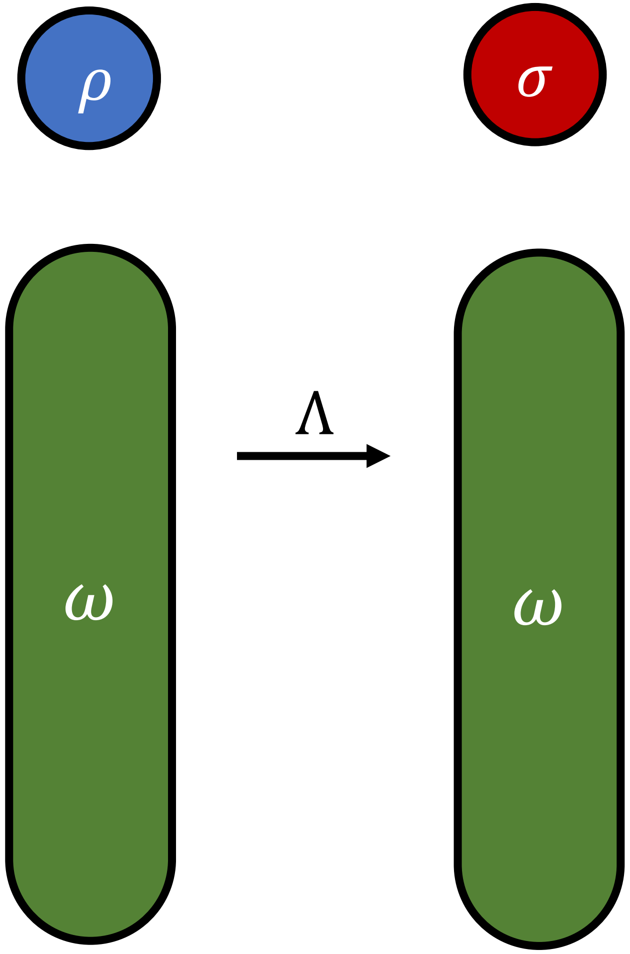
Example: Pure-state entanglement (LOCC)

$$R_\epsilon^\infty(\psi,\phi) \coloneqq \lim_{n\to\infty} R_\epsilon^n(\psi,\phi)$$

$$= \frac{H(\psi_A)}{H(\phi_A)}$$

C. Bennett et al, **Phys. Rev. A** (1996)

Catalytic transformations

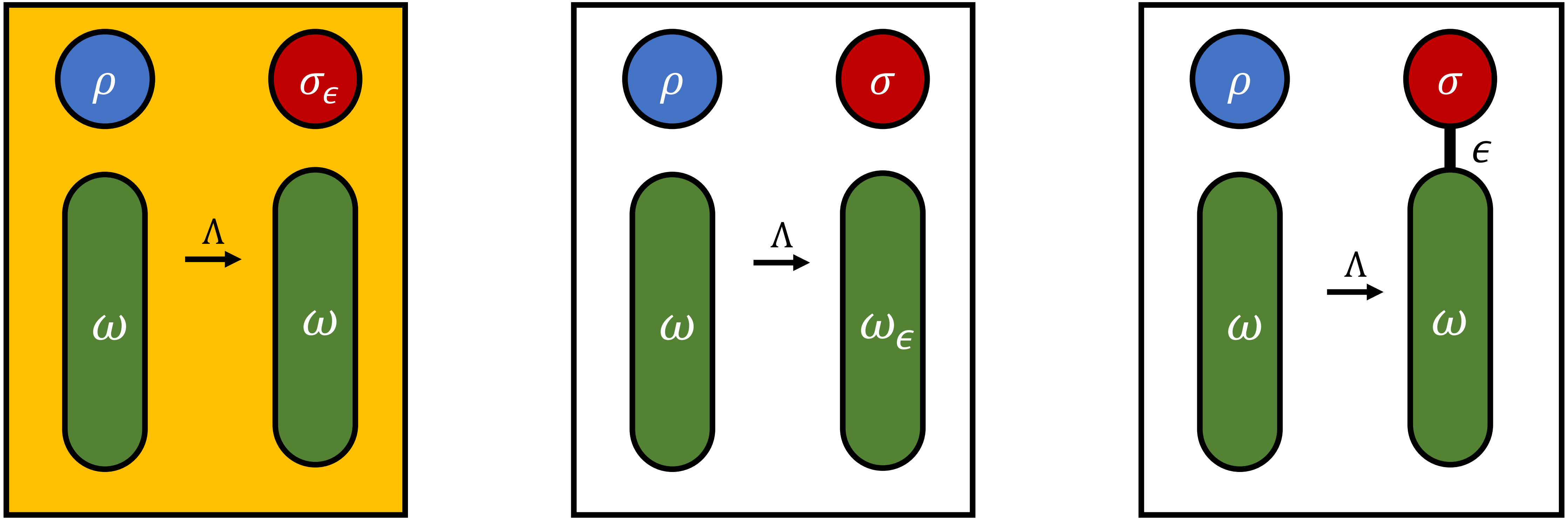


$$\Lambda(\rho \otimes \omega) \approx \sigma \otimes \omega$$

Basic question: When is this possible?

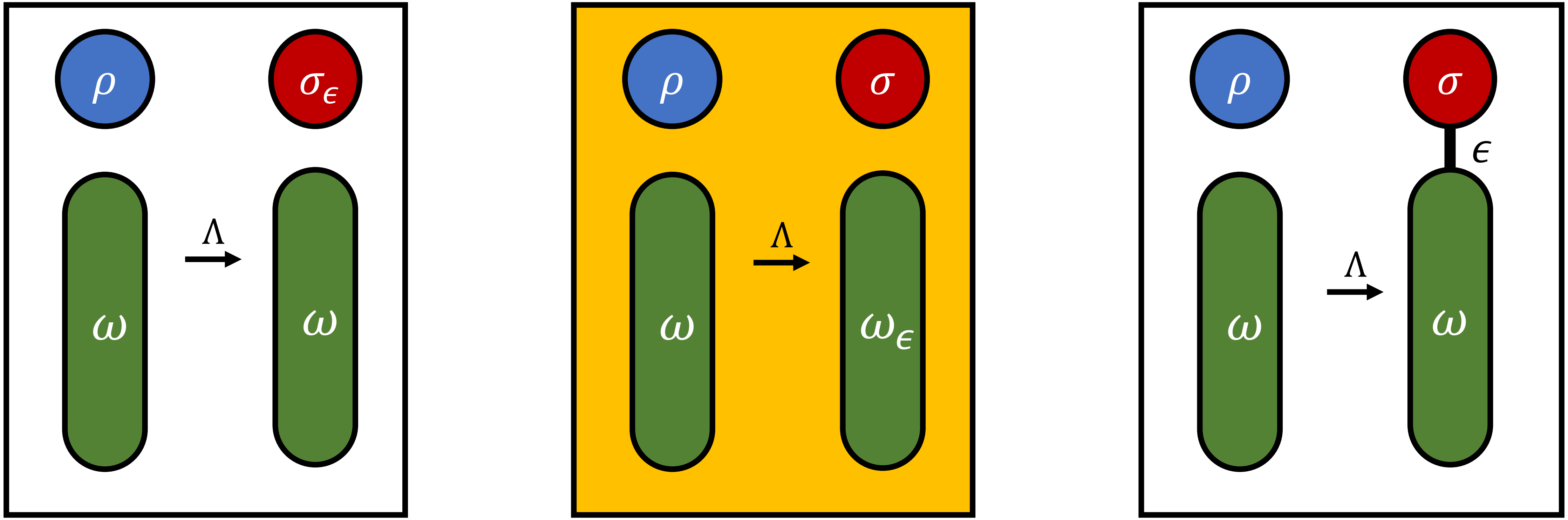
D. Jonathan, M. B. Plenio, **Phys. Rev. Lett.** (1999)

Catalytic transformations



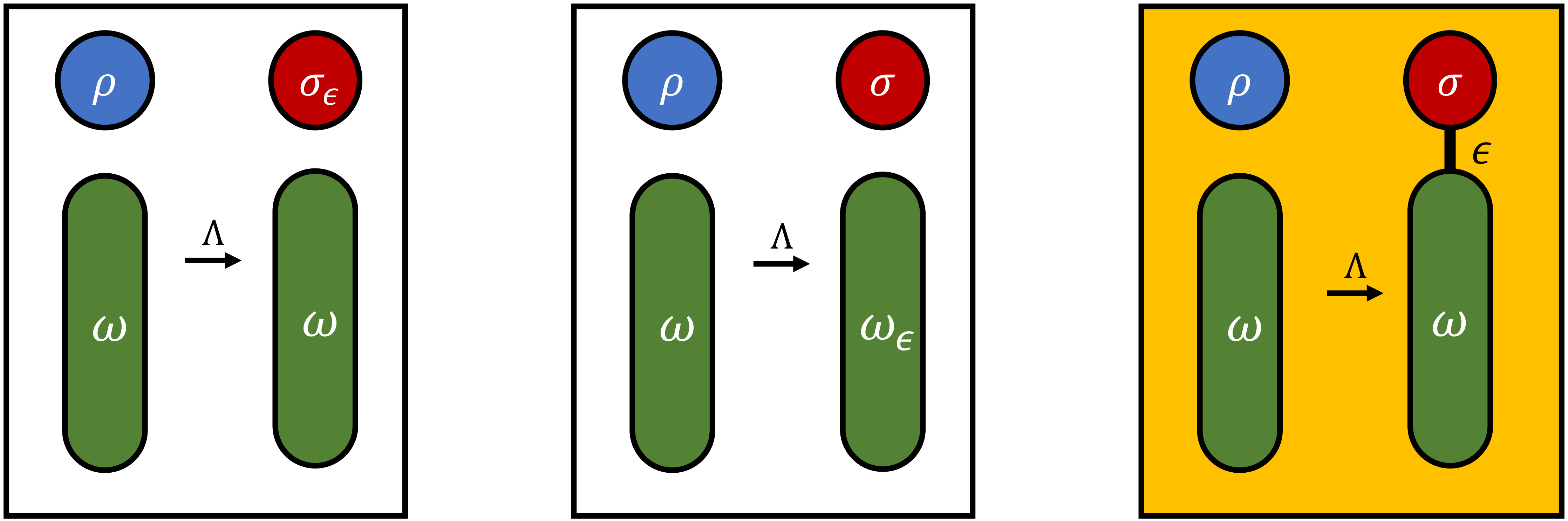
D. Jonathan, M. B. Plenio, **Phys. Rev. Lett.** (1999)

Catalytic transformations



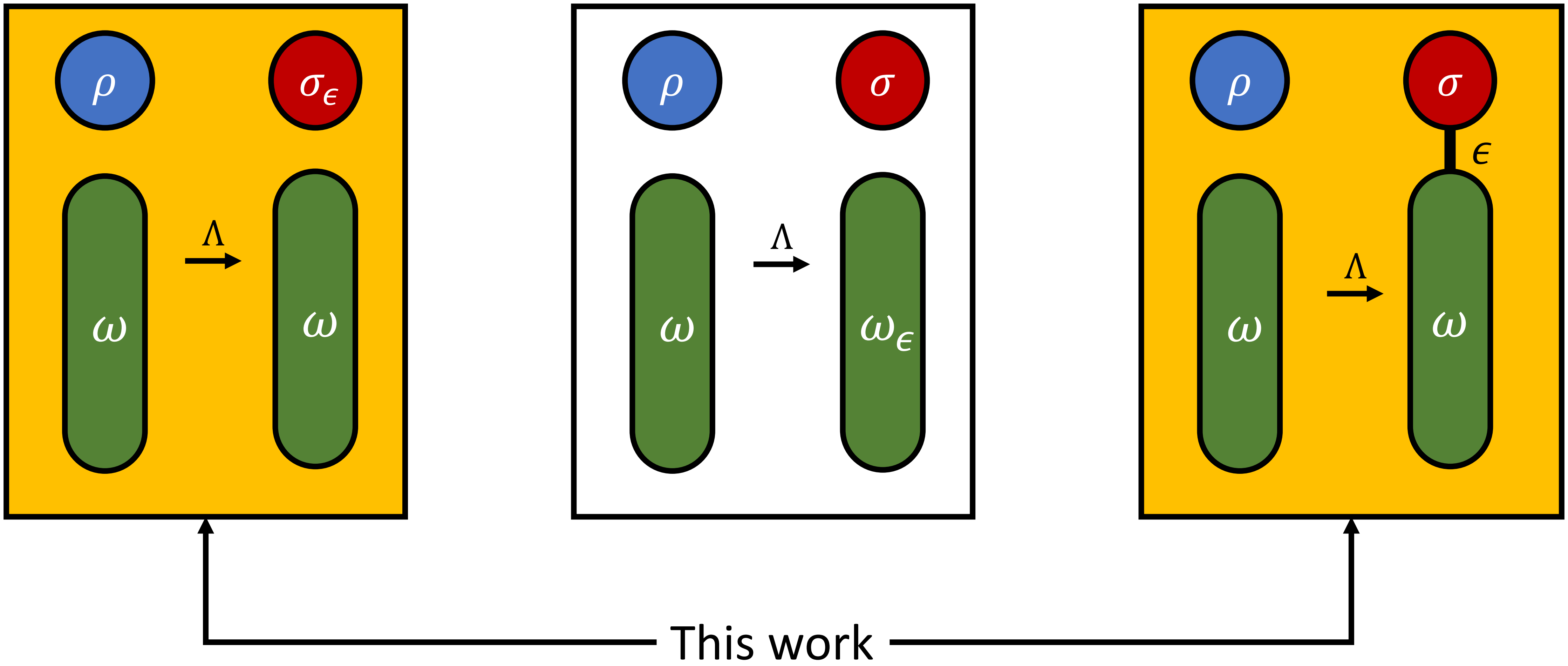
F. Brandao et al., ***PNAS*** (2015)

Catalytic transformations



R. Gallego et. al., **New J. Phys.** (2016)

Catalytic transformations



R. Gallego et. al., **New J. Phys.** (2016)

Catalytic transformations

Correlated-catalytic free transformation between ρ_S and σ_S exists if for any $\epsilon > 0$ there is a catalytic system C in a state ω_C and a free operation Λ so that

$$\Lambda[\rho_S \otimes \omega_C] = \eta_{SC}$$

where

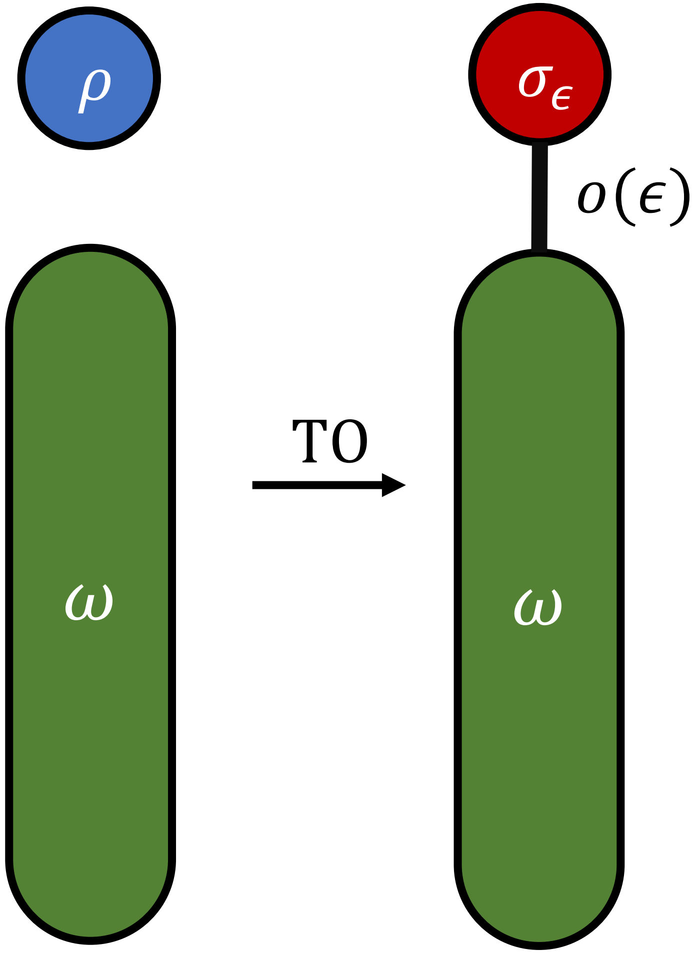
catalytic $\qquad \eta_C = \omega_C$

approximate $\qquad \|\eta_S - \sigma_S\| < \epsilon$

weak correlations $\qquad \|\eta_{SC} - \sigma_S \otimes \omega_C\| < o(\epsilon)$

$$\rho_S \xrightarrow[\mathcal{F}]{\epsilon} \sigma_S$$

Catalytic transformations



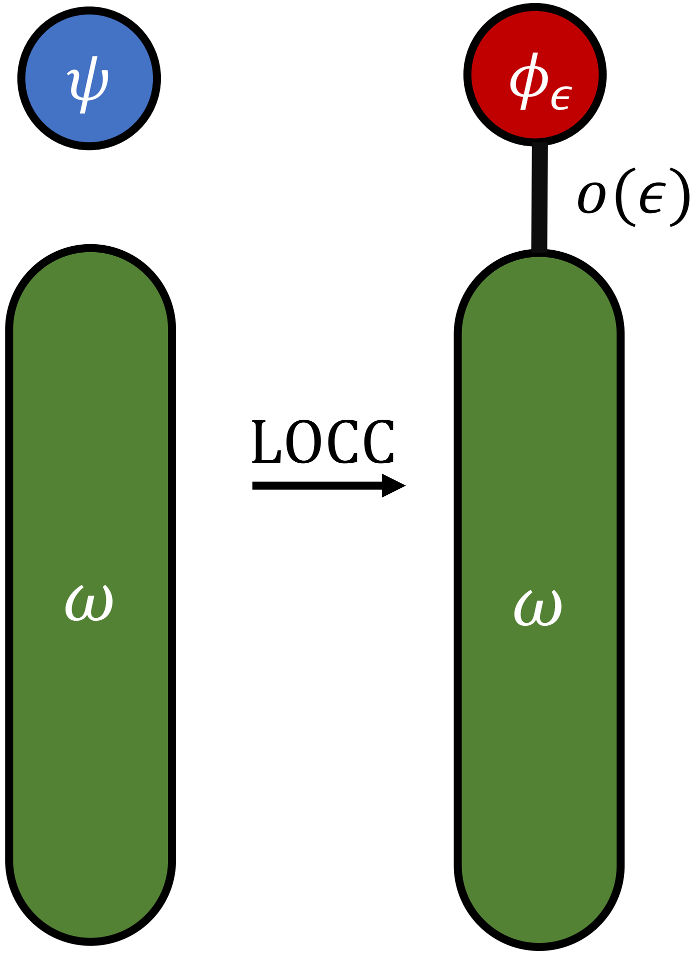
$$\rho \xrightarrow[\text{TO}]{\epsilon} \sigma$$

Example: Thermodynamics (thermal operations)

$$D(\rho||\gamma) > D(\sigma||\gamma) \qquad \gamma \propto e^{-\beta H}$$

M. P. Müller, **Phys. Rev. X** (2018)
N. Shiraishi, T. Sagawa, **Phys. Rev. Lett.** (2021)

Catalytic transformations



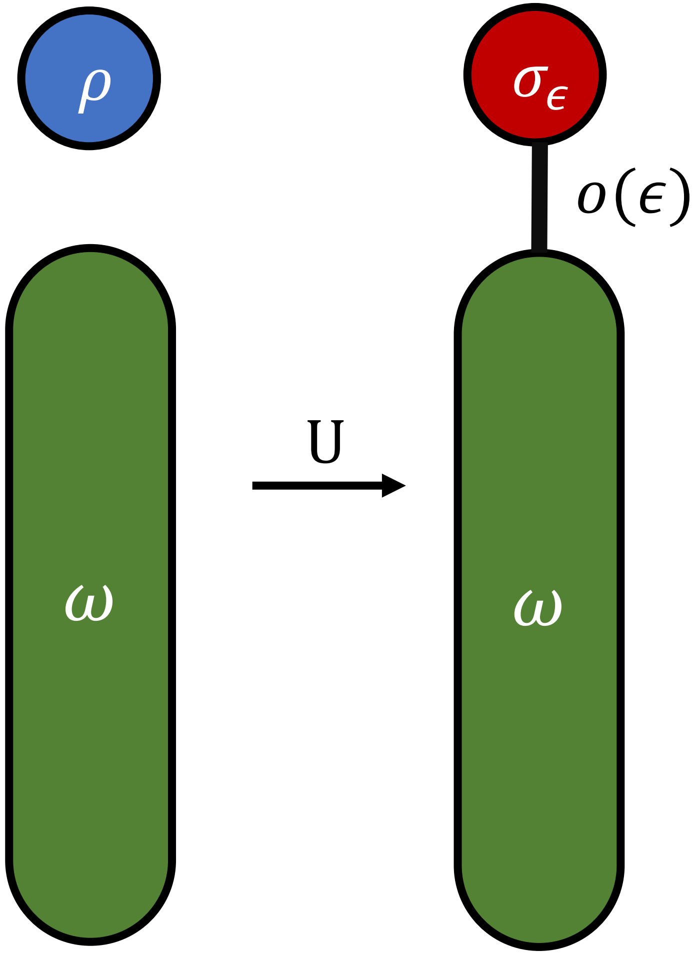
$$\psi \xrightarrow[\text{LOCC}]{\epsilon} \phi$$

Example: Entanglement (LOCC)

$$H(\psi_A) > H(\phi_A)$$

T. V. Kondra et. al., **Phys. Rev. Lett.** (2021)

Catalytic transformations



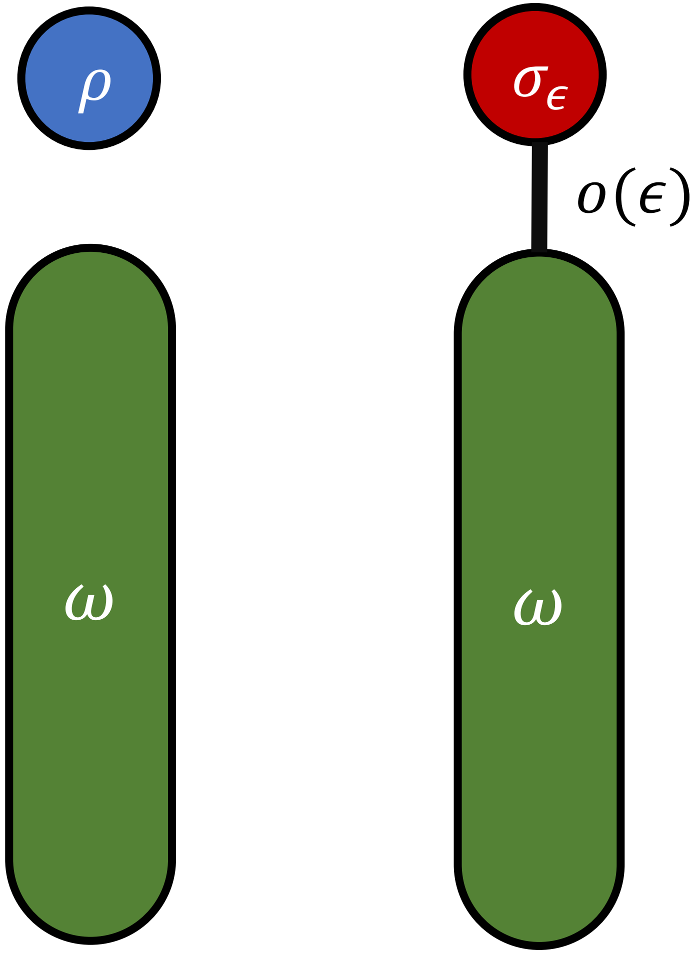
$$\rho \xrightarrow[\text{U}]{\epsilon} \sigma$$

Example: Quantum mechanics (unitaries)

$$H(\rho) > H(\sigma)$$

H. Wilming, **Phys. Rev. Lett**. (2021)

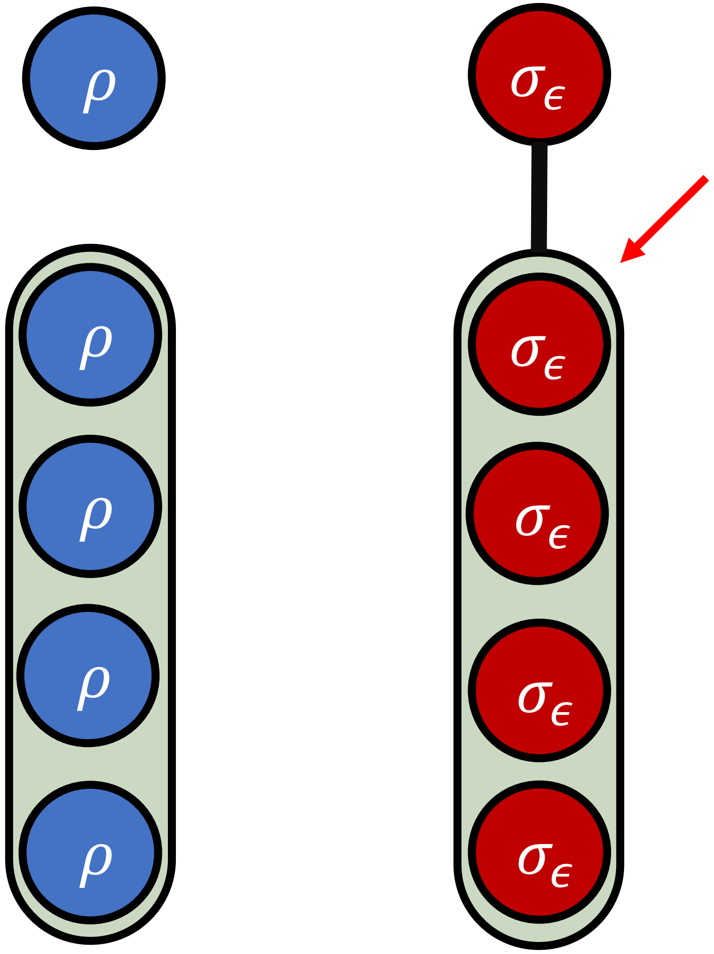
Catalytic transformations



$$\rho \xrightarrow[\mathcal{F}]{\epsilon} \sigma$$

Problem: What about the size of the catalyst?

Catalytic transformations



$$\Lambda(\rho \otimes \omega) \approx \sigma \otimes \omega \quad \Leftrightarrow \quad D(\rho||\gamma) > D(\sigma||\gamma)$$

requires infinitely large catalysts

Problem: What about the size of the catalyst?

Example: Thermodynamics (thermal operations)

$$D(\rho||\gamma) \approx D(\sigma||\gamma) \longrightarrow d_C \approx \infty$$

P. Boes et. al., **PRX Quantum** (2022)

Results

Suppose the rate $R_\epsilon^n(\rho,\sigma)$ can be written as

$$R_\epsilon^n(\rho,\sigma) = R^\infty(\rho,\sigma) - \frac{1}{\sqrt{n}} R'_\epsilon(\rho,\sigma) - o\left(\frac{1}{\sqrt{n}}\right).$$

Examples:

- Entanglement (LOCC), W. Kumagai and M. Hayashi, **IEEE Trans. Inf. Theory** (2017)
- Thermodynamics (thermal operations), C. Chubb et. al., **Quantum** (2018)
- Coherence (incoherent operations), C. Chubb et. al., **Quantum** (2018)
- Classical communication (quantum channels). C. Chubb et. al., **Comm. Math. Phys.** (2017)

Main Result

If $R^{\infty}(\rho, \sigma) > 1$ then for all $\epsilon > 0$ there exists a correlated-catalytic free operation

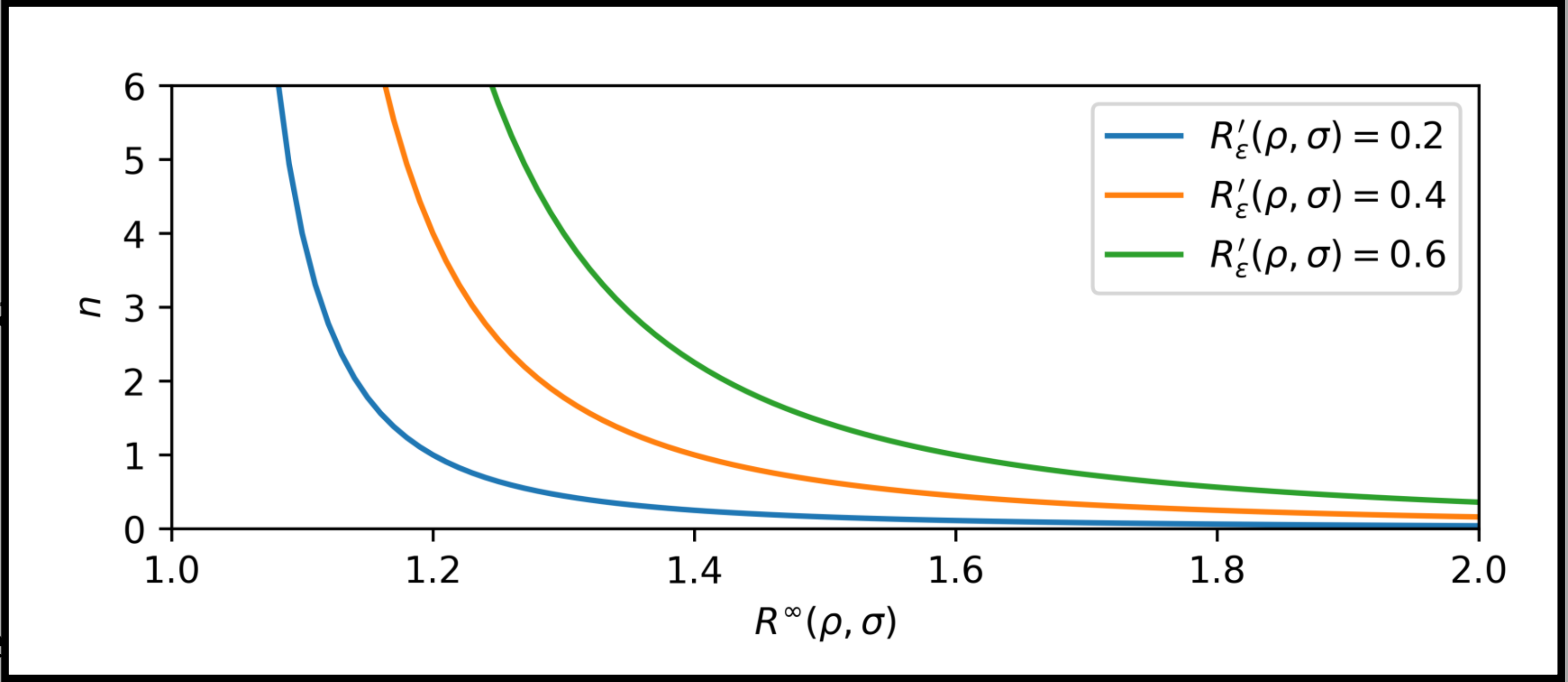
$$\rho \xrightarrow[\mathcal{F}]{\epsilon} \sigma$$

with a catalyst C of dimension $d = e^{O(n)}$ where

$$n = \left(\frac{R'_{\epsilon}(\rho, \sigma)}{R^{\infty}(\rho, \sigma) - 1} \right)^2$$

Main Result

If $R^{\infty}(\rho,\sigma) > 1$ then for
operation



with a catalyst C of dime

$$n = \left(\frac{R'_{\epsilon}(\rho,\sigma)}{R^{\infty}(\rho,\sigma) - 1}\right)^2$$

Application: Thermodynamics (thermal operations)

Let ρ and σ be such that $[\rho, H] = 0$ and $[\sigma, H] = 0$. Assume for some $\epsilon > 0$ we have

$$D(\rho||\gamma) > D(\sigma||\gamma).$$

Then $\rho \xrightarrow[\mathrm{TO}]{\epsilon} \sigma$.

M. P. Müller, **Phys. Rev. X** (2018)

Application: Thermodynamics (thermal operations)

Let ρ and σ be such that $[\rho, H] = 0$ and $[\sigma, H] = 0$. Assume for some $\epsilon > 0$ we have

$$D(\rho||\gamma) > D(\sigma||\gamma) + \frac{c(\epsilon)}{\sqrt{\log d}}$$

Then $\rho \xrightarrow[\mathrm{TO}]{\epsilon} \sigma$ with a catalyst of dimension d.

Application: Thermodynamics (thermal operations)

Let ρ and σ be such that $[\rho, H] \neq 0$ and $[\sigma, H] = 0$. Assume for some $\epsilon > 0$ we have

$$D(\rho||\gamma) > D(\sigma||\gamma) + \frac{c'(\epsilon)}{\sqrt{\log d}}$$

Then $\rho \xrightarrow[\text{TO}]{\epsilon} \sigma$ with a catalyst of dimension d (sufficiently large).

Application: Thermodynamics (thermal operations)

$$D(\rho||\gamma) > D(\sigma||\gamma) + \frac{c(\epsilon)}{\sqrt{\log d}}$$

Application: Thermodynamics (thermal operations)

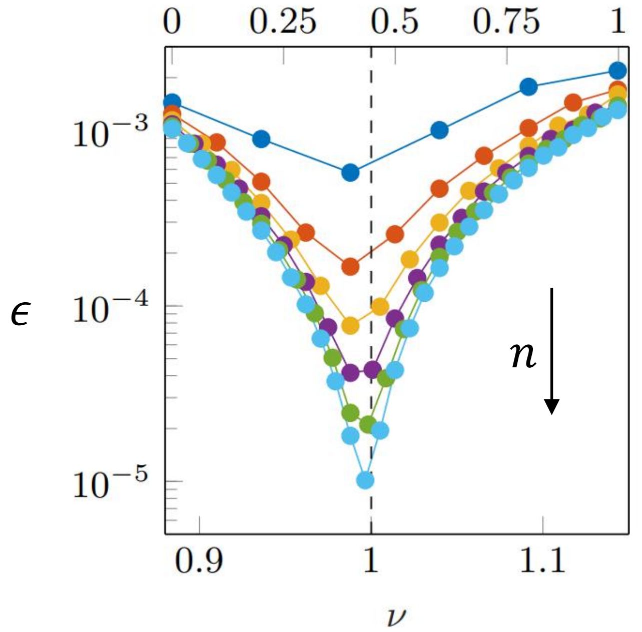
$$D(\rho||\gamma) > D(\sigma||\gamma) + \frac{c(\epsilon)}{\sqrt{\log d}}$$ → **Resonance:** $c(\epsilon) \to 0$ for any $\epsilon > 0$

Resonance in non-asymptotic transformations

$$R^n_\epsilon(\rho,\sigma) = R^\infty(\rho,\sigma) - \frac{1}{\sqrt{n}} R'_\epsilon(\rho,\sigma) - o\left(\frac{1}{\sqrt{n}}\right).$$

$$R'_\epsilon(\rho,\sigma) \to 0 \quad \Leftrightarrow \quad v \to 1$$

$$v := \frac{V(\rho||\gamma)/D(\rho||\gamma)}{V(\sigma||\gamma)/D(\sigma||\gamma)}$$



K. Korzekwa et. al., **Phys. Rev. Lett.** (2019)

Application: Thermodynamics (thermal operations $\beta = 0$)

$$D(\rho||\gamma) > D(\sigma||\gamma) + \frac{c(\epsilon)}{\sqrt{\log d}}$$

Application: Thermodynamics (thermal operations $\beta = 0$)

$$H(\sigma) > H(\rho) + \frac{c(\epsilon)}{\sqrt{\log d}}$$

Application: Thermodynamics (thermal operations $\beta = 0$)

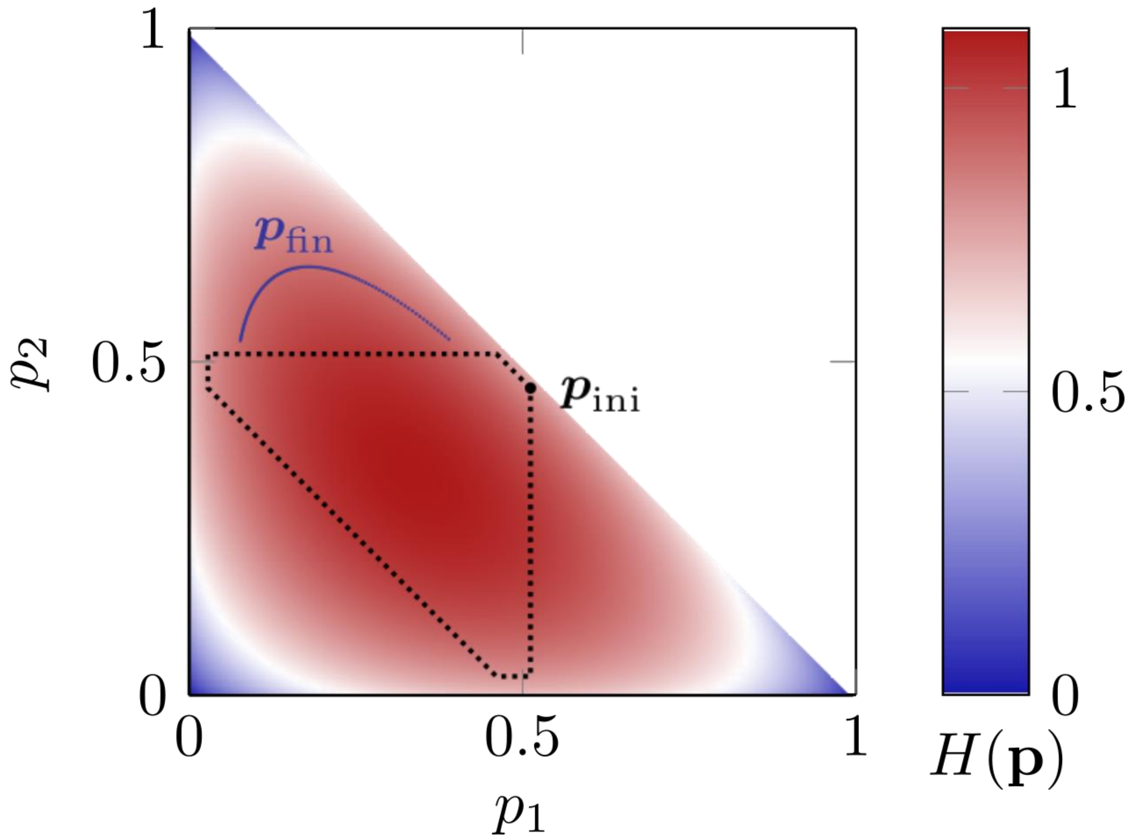
$$H(\sigma) > H(\rho) + \frac{c(\epsilon)}{\sqrt{\log d}}$$

Resonance: $c(\epsilon) \to 0$ for any $\epsilon > 0$

Initial: $\rho = \mathrm{diag}(\boldsymbol{p}_{\mathbf{ini}})$

Final: a family $\{\sigma_k\}$ with $\sigma_k = \mathrm{diag}(\boldsymbol{p}_{\mathbf{fin}})$
with $H(\sigma_k || \gamma) = \mathrm{const}$

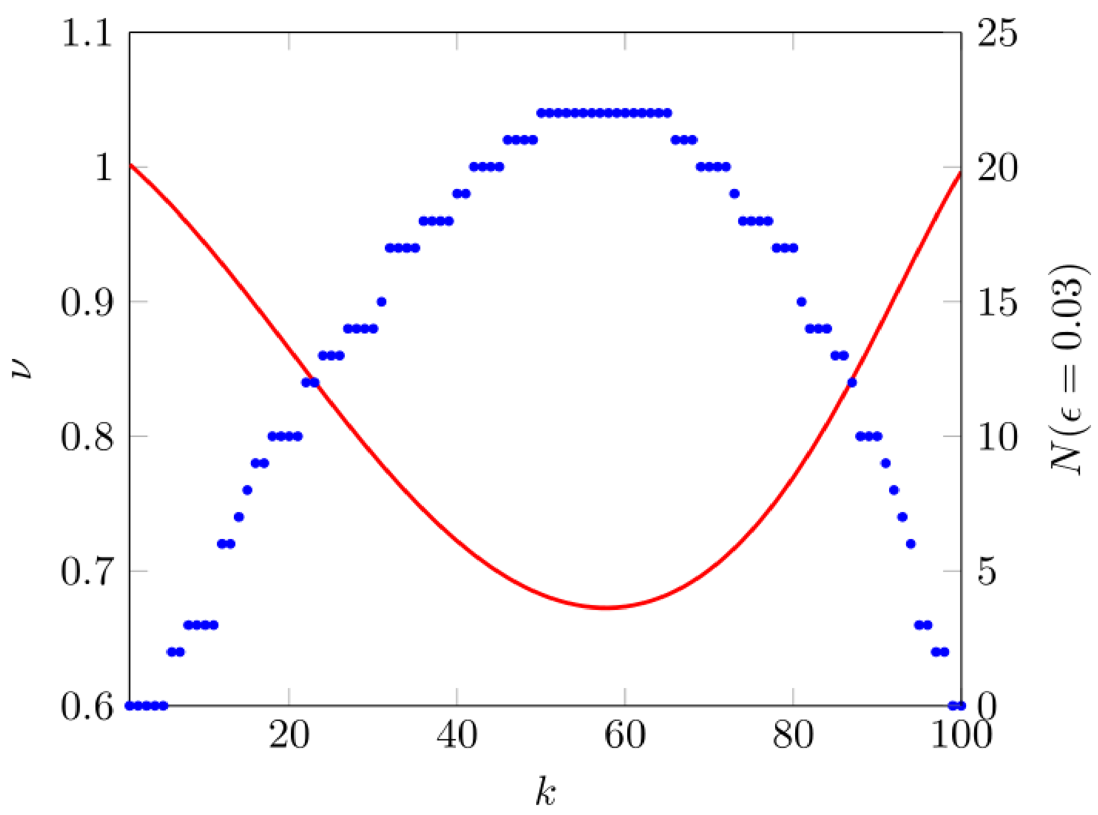
Goal: Perform $\rho \xrightarrow[\mathrm{TO}]{\epsilon} \sigma_k$

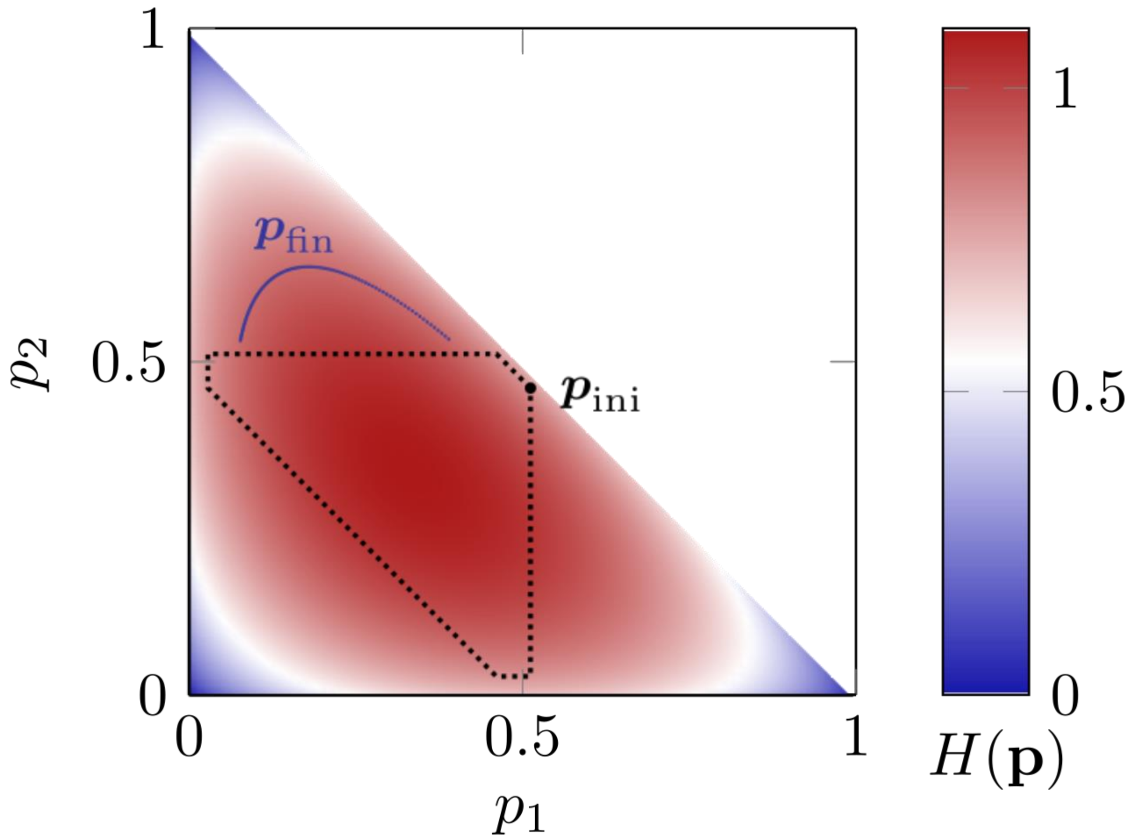


Application: Thermodynamics (thermal operations $\beta = 0$)

$$H(\sigma) > H(\rho) + \frac{c(\epsilon)}{\sqrt{\log d}}$$

Resonance: $c(\epsilon) \to 0$ for any $\epsilon > 0$





Take-home messages

Correlated-catalytic operations “emulate” collective operations

Collective effects (e.g. resonance) have counterparts in catalytic operations

Finite-size corrections to asymptotic rates imply bounds on the catalyst’s size

Thank you!

Resonance

$$R_\epsilon^\infty(\rho,\sigma) = \frac{D(\rho||\gamma)}{D(\sigma||\gamma)}, \qquad R'_\epsilon(\rho,\sigma) = \sqrt{\frac{D(\rho||\gamma)V(\sigma||\gamma)}{D(\sigma||\gamma)^3}}\, f_v(\epsilon), \qquad v := \frac{V(\rho||\gamma)/D(\rho||\gamma)}{V(\sigma||\gamma)/D(\sigma||\gamma)}$$

$$\rho(\lambda) = \rho_1^{\otimes \lambda n} \otimes \rho_2^{\otimes (1-\lambda)\, n}$$

$$\mathrm{LOCC}\big(\rho(\lambda)\big) \approx_a \sigma^{\otimes n}$$

$c(\epsilon) \to 0$ is equivalent to having $f_v(\epsilon) \to 0$

$c(\epsilon) \to 0$ **for any** $\epsilon > 0$

Application: Thermodynamics (thermal operations)

Let ρ and σ be such that $[\rho, H] \neq 0$ and $[\sigma, H] = 0$. Assume for some $\epsilon > 0$ we have

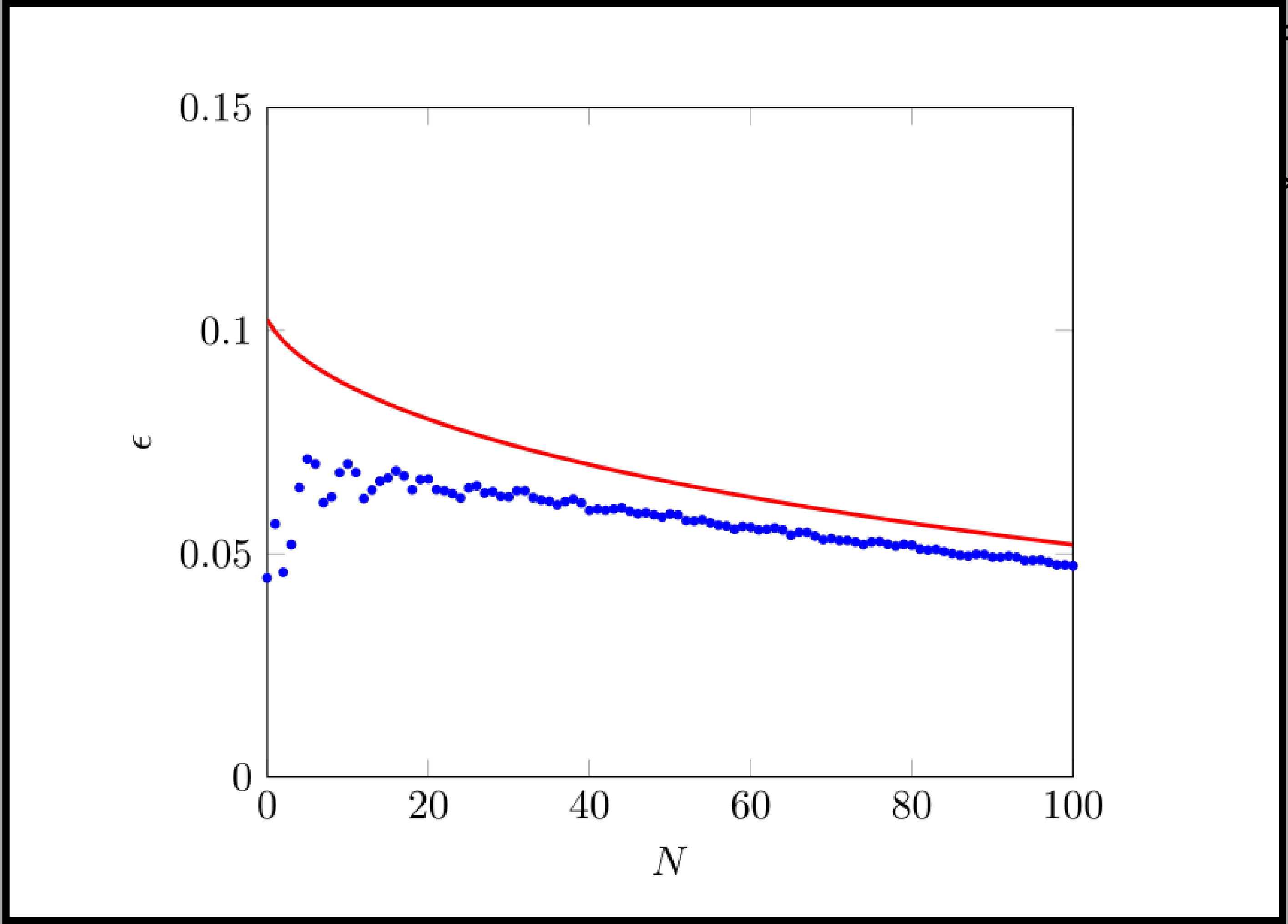
$$D(\rho||\gamma) - D(\sigma||\gamma) > \frac{c(\epsilon)}{\sqrt{\log d}} + o(1)$$

Then $\rho \xrightarrow[\text{TO}]{\epsilon} \sigma$ with a catalyst of dimension d.

$$D(\rho||\gamma) - D(\sigma||\gamma) > \frac{c'(\epsilon)}{\sqrt{\log d}}.$$

Suppose we can prepare any catalyst of a fixed dimension $d = e^{O(n)}$.
What is the error ϵ of our transformation?

Application: Thermodynamics (thermal operations)



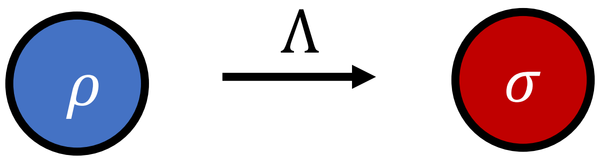
= 0. Assume for some $\epsilon > 0$ we have

$$> \frac{c(\epsilon)}{\sqrt{\log d}} + o(1).$$

$$D(\rho||\gamma) - D(\sigma||\gamma) > \frac{c'(\epsilon)}{\sqrt{\log d}}.$$

$$D(\rho||\gamma) - D(\sigma||\gamma) > \frac{c(\epsilon)}{\sqrt{\log d}}.$$

State transformations



$$\Lambda(\rho) \approx \sigma$$

$\Lambda \in \mathcal{F}$ (free operations)

Basic question: When is this possible?

M. Horodecki, J. Oppenheim, **Int. J. Mod. Phys. B** 1345019 (2013)

E. Chitambar, G. Gour, **Rev. Mod. Phys.** 91, 025001 (2019)

Basic Lemma

Suppose $R_\epsilon^n(\rho, \sigma) > 1$ for some $\epsilon > 0$ and some $n \in \mathbb{N}$. Then

$$\rho_S \xrightarrow[\mathcal{F}]{\epsilon} \sigma_S$$

with a catalyst C of dimension d satisfying

$$\log d = O(n)$$

Application: Thermodynamics (thermal operations)

Let ρ and σ be such that $[\rho, H] \neq 0$ and $[\sigma, H] = 0$. Assume for some $\epsilon > 0$ we have

$$D(\rho||\gamma) > D(\sigma||\gamma) + \frac{c(\epsilon)}{\sqrt{\log d}} + o(1)$$

Then $\rho \xrightarrow[\text{TO}]{\epsilon} \sigma$ with a catalyst of dimension d.

$$D(\rho||\gamma) - D(\sigma||\gamma) > \frac{c'(\epsilon)}{\sqrt{\log d}}.$$