QUANTIFYING QUBIT MAGIC RESOURCE WITH GOTTESMAN-KITAEV-PRESKILL ENCODING

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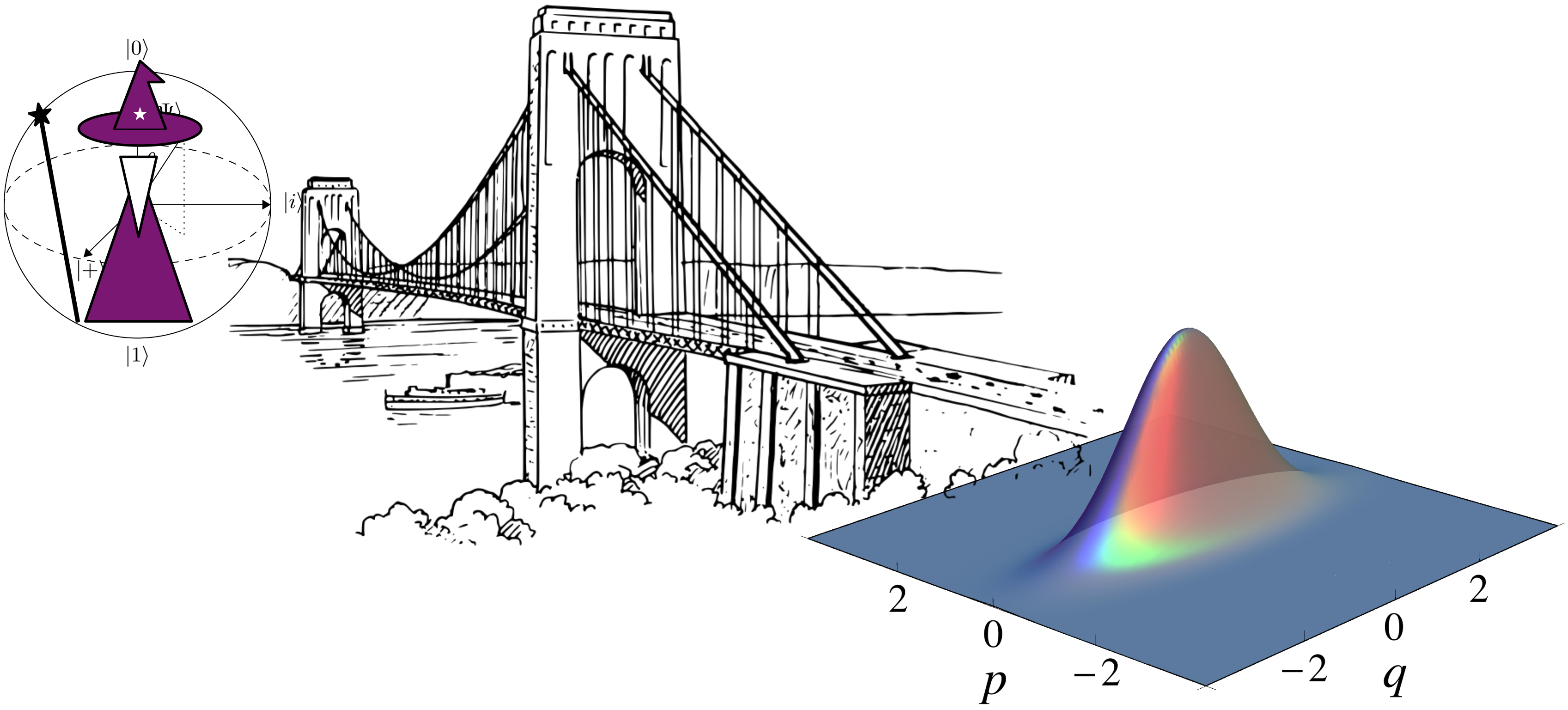

$|0\rangle$
$|i\rangle$
$|1\rangle$
2
0
p
−2
2
0
−2
q

QUANTUM COMPUTING WITH QUBITS

- We consider an array of two-level systems (qubits) $|\psi\rangle = \sum_{\boldsymbol{i}\in\mathcal{F}_2^n} c_{\boldsymbol{i}} |\boldsymbol{i}\rangle$

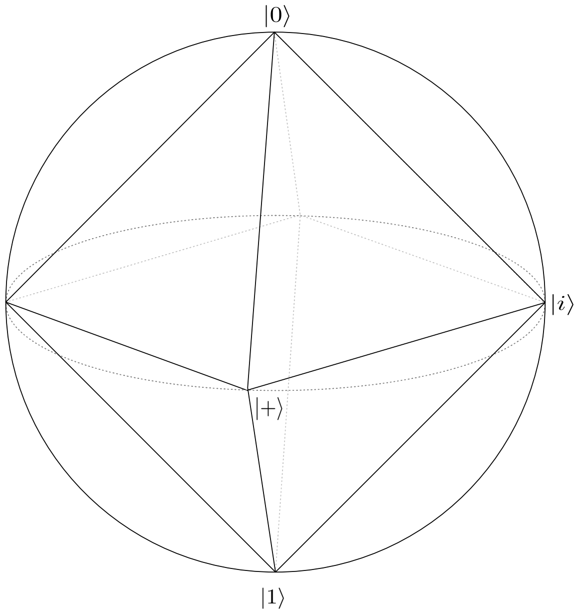
- Operations are implemented using quantum gates

- A Universal gate set is $\{\underbrace{CX, H, S}_{\text{Clifford}}, T\}$

$$C_2 \equiv \{\hat{U} | \hat{U}\hat{P}\hat{U}^\dagger \subseteq C_1, \forall \hat{P} \in C_1\}$$

STABILIZER AND MAGIC STATES

- Stabilizer states are closed under Clifford
- Magic states enable non-Clifford operations through teleportation



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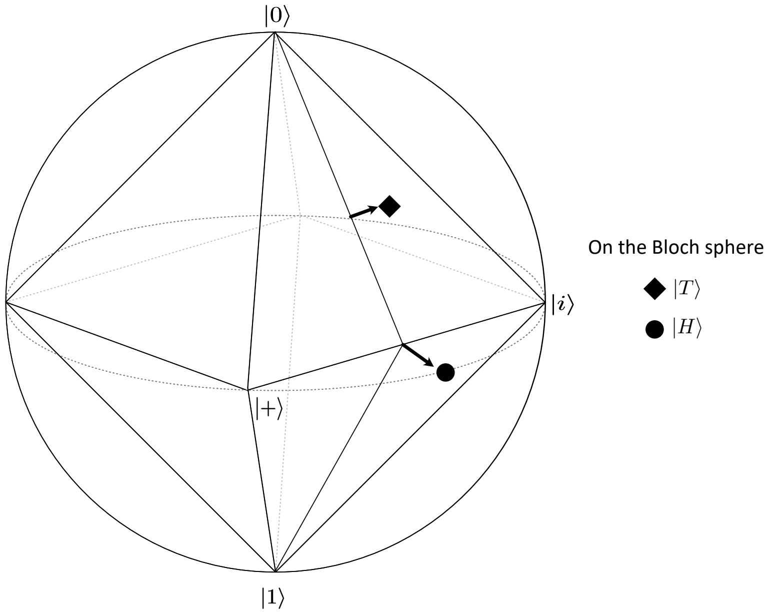


$$|H\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$$

$$|T\rangle = \cos(\beta)|0\rangle + \sin(\beta)e^{i\frac{\pi}{4}}|1\rangle$$

$$\cos(2\beta) = \frac{1}{\sqrt{3}}$$

MAGIC AND STABILISER STATES

- Gottesman-Knill theorem:

A quantum computer based only on:

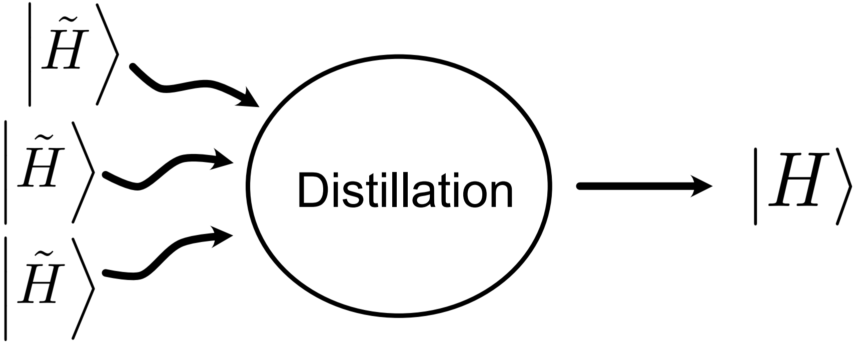
1. Qubits initialzed in a Pauli eigenstate
2. Clifford group operations
3. Pauli measurements

Can be simulated efficiently with a classical computer

D. Gottesman, arXiv:quant-ph/9807006 (1998)

RESOURCES IN FAULT-TOLERANCE

- 2D topological codes: Only Clifford operations are natively fault-tolerant
- Magic-state distillation is a way for non-Clifford gates



- Requires most of quibts/gates used

Resource: Magic/ non-stabilizerness

S. Bravyi, et al, PRA 71 022316 (2005)

MAGIC MEASURES

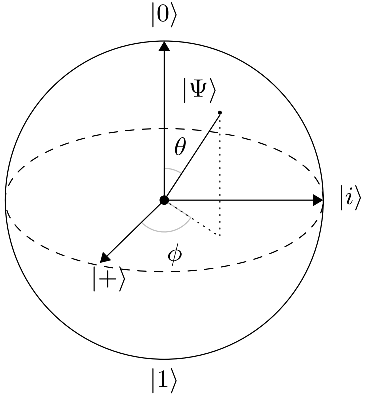
- Relative entropy of magic, *V. Veitch et al, New Journal of Physics 16, 013009 (2014)*
- Robustness of magic, *Howard et al, Phys. Rev. Lett. 118, 090501 (2017)*
- Stabilizer rank, *Bravyi et al, Quantum 3, 181 (2019)*
- Stabilizer nullity and the dyadic monotone, *Beverland et al, Quantum Science and Technology 5, 035009 (2020)*
- Dyadic negativity, mixed state extend, generalised robustness, *Seddon et al, PRX Quantum 2, 010345 (2021)*

- Stabilizer Rény Entropy, *Leone et al, PRL 128, 050402 (2022)*
- Bell magic, *Haug et al, arXiv:2204.10061 (2022)*

DISCRETE VS CONTINUOUS

Discrete variables:

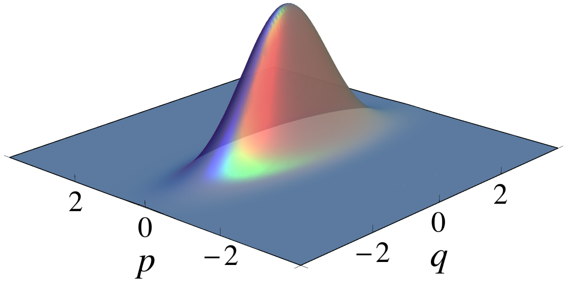
- Discrete basis
- Information encoded in Qubits
- Finite dimensional Hilbert space



Bloch Sphere

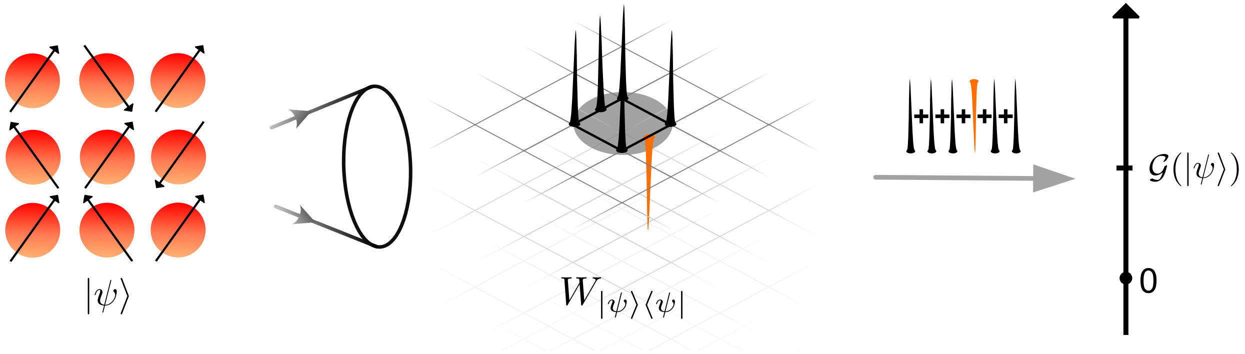
Continuous variables:

- Information encoded in q. modes
- Relevant observables have continuous spectra
- Infinite dimensional Hilbert space



Phase space representation

NEW MEASURE



N-qubit state

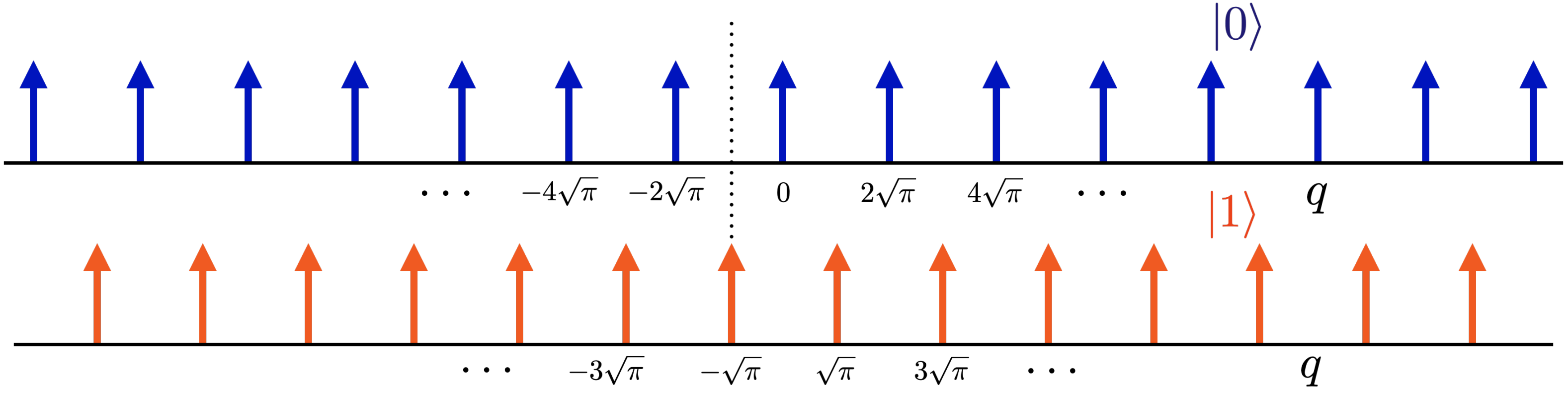
GKP Wigner function

WLN per cell
Subtract inherent negativity

GOTTESMAN-KITAEV-PRESKILL CODE

- Error correction code for continous variable systems to encode qubits

$$|u_i\rangle = \sum_{s_i=-\infty}^{\infty} \left|x_i = \sqrt{\pi}(u_i + 2s_i)\right\rangle_{\hat{q}}$$



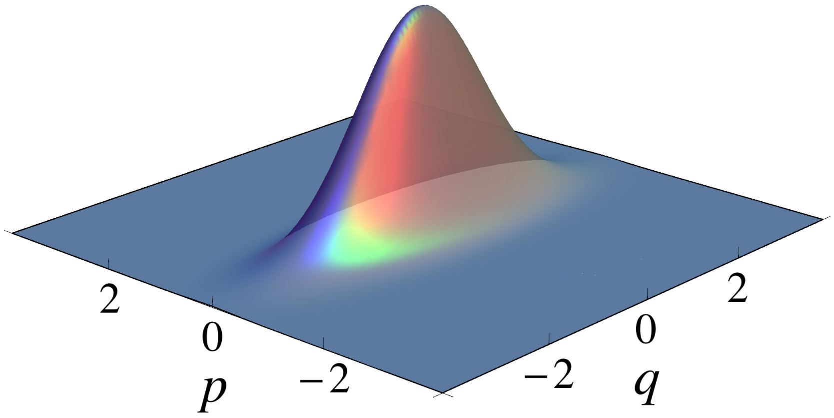
D. Gottesman, et al, Phys. Rev. A 64, 012310 (2001)

WIGNER FUNCTION

- Wigner Function represents a state in a phase-space

$$W_{\hat{\rho}}(\boldsymbol{q},\boldsymbol{p}) \equiv \frac{1}{(2\pi)^n}\int \mathrm{d}^n\boldsymbol{x}\, e^{i\boldsymbol{p}\boldsymbol{x}} \left\langle \boldsymbol{q}+\frac{\boldsymbol{x}}{2}\right|\hat{\rho}\left|\boldsymbol{q}-\frac{\boldsymbol{x}}{2}\right\rangle_{\hat{q}}$$

- Is Quasi-probability distribution
- Resource theory related to Wigner negativity



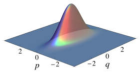
WIGNER LOGARITHMIC NEGATIVITY

- Resource theory of Wigner logarithmic negativity (WLN)
  - Gaussian states are the free states
  - Gaussian operations are the free operations

$$\mathcal{W}(\hat{\sigma}) = \log_2 \left(\int \mathrm{d}^n \boldsymbol{q} \mathrm{d}^n \boldsymbol{p} |W_{\hat{\sigma}}(\boldsymbol{q}, \boldsymbol{p})| \right)$$

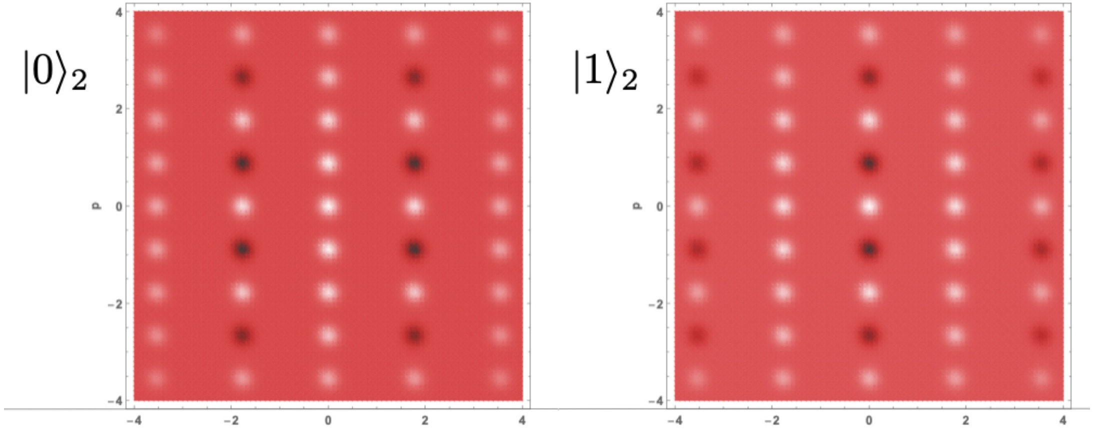
- Feature of GKP: All Clifford operations are Gaussian
- Transfer properties using GKP to define our measure

F. Albarelli, et al, Phys. Rev. *A* 98.5, 052350 (2018)
R. Takagi, et al, Phys. Rev. A 97, 062337 (2018)



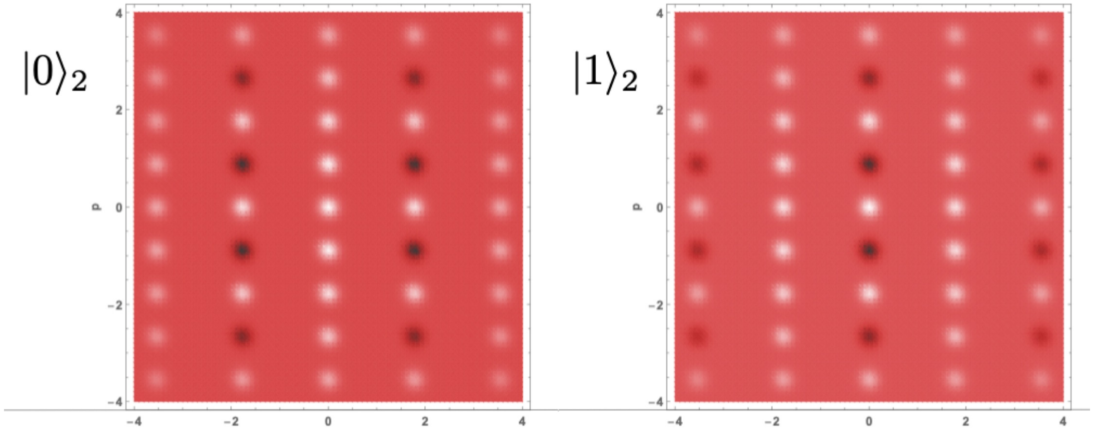
WIGNER FUNCTION: GKP STATE

- Stabilizer states: ¼ of peaks are negative

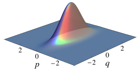


WIGNER LOG. NEGATIVITY: GKP STATE

- WLN is infinite negativity even for stabilizer states



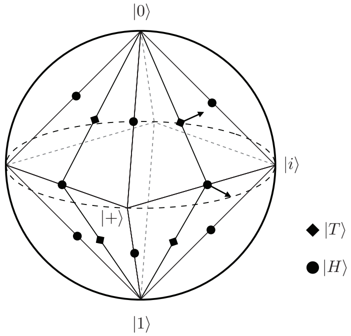
- Lattice: We restrict to one unit cell
- WLN is non-zero for stabilizer states
  - But constant!

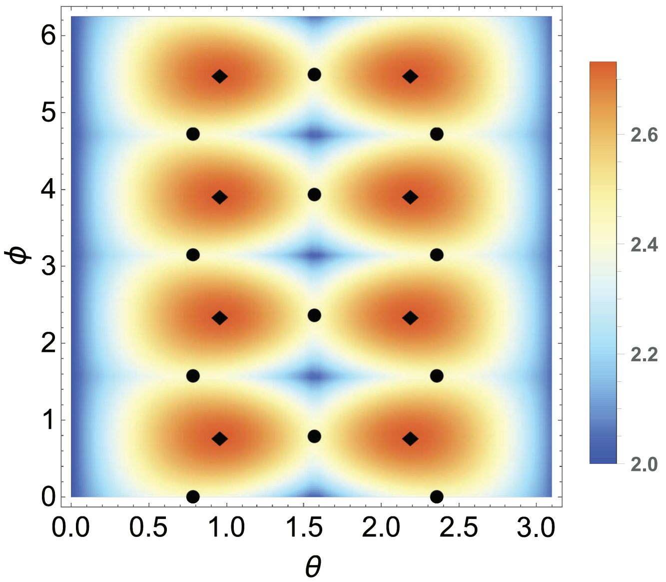


WIGNER LOG. NEGATIVITY: GKP STATE

- WLN has constant value for stabiliser states
- WLN has maximal values for T/H-type states

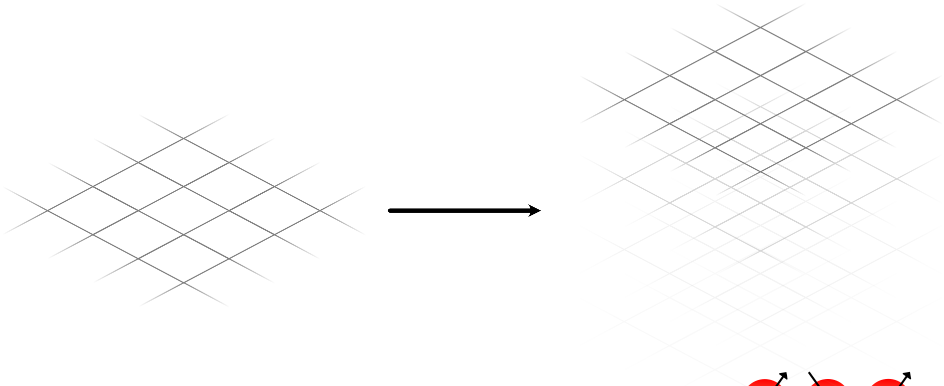
$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

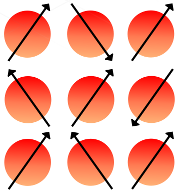




L. García-Álvarez, et . al, *Int. Symposium on Mathematics, Quantum Theory, and Cryptography*, 2021

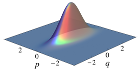
EXTENSION TO N-QUBITS



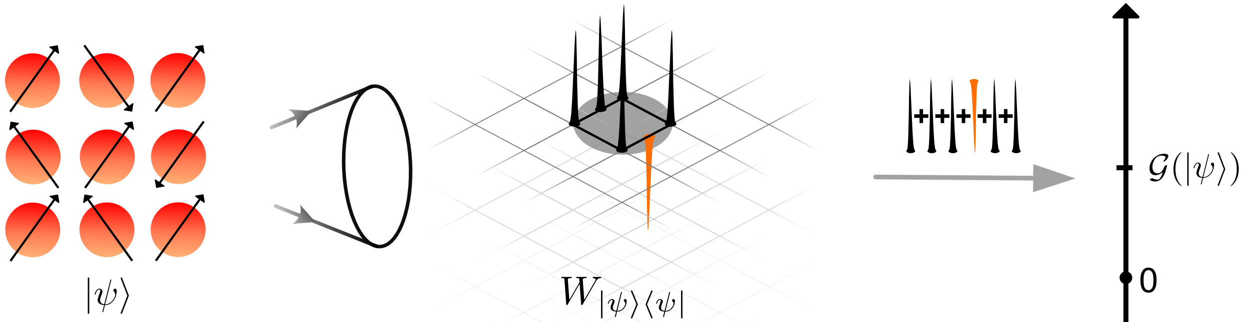


$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$|\psi\rangle = \sum_{\boldsymbol{i}\in\mathcal{F}_2^n} c_{\boldsymbol{i}}\,|\boldsymbol{i}\rangle$$



NEW MEASURE



N-qubit state

$$|\psi\rangle = \sum_{i\in\mathbb{F}_2^n} c_i\,|i\rangle$$

GKP Wigner function

WLN per cell

$$\mathcal{W}_C(\hat{\rho}) = \log_2\left(\int_0^{2\sqrt{\pi}-\epsilon} \mathrm{d}^n\boldsymbol{q}\,\mathrm{d}^n\boldsymbol{p}\,|W_{\hat{\rho}}(\boldsymbol{q},\boldsymbol{p})|\right)$$

Subtract inherent negativity

THE MEASURE

$$\mathcal{G}(|\psi\rangle) = \log_2 \left(\sum_{\boldsymbol{i},\boldsymbol{j} \in \mathbb{F}_2^n} \left| \sum_{\boldsymbol{k} \in \mathbb{F}_2^n} \frac{(-1)^{\boldsymbol{i}\cdot\boldsymbol{k}}}{2^n} c_{\boldsymbol{k}} c_{\boldsymbol{k}+\boldsymbol{j}} \right| \right)$$

- No obvious connection to continuous variables left
- We recover the st-norm!
  - And prove that it is a magic measure for pure states
  - And the stabilizer Rényi entropy for α= ½
- Easier to compute than other measures (up to 12 qubits on a laptop)

THE MEASURE

$$\mathcal{G}(|\psi\rangle) = \log_2\left[\frac{1}{2^n}\sum_{\hat{P}\in\mathcal{P}_n^+}\left|\mathrm{Tr}\left[\hat{P}\,|\psi\rangle\langle\psi|\right]\right|\right]$$

- No obvious connection to continuous variables left
- We recover the st-norm!
  - And prove that it is a magic measure for pure states
  - And the stabilizer Rényi entropy for $\alpha = 1/2$
- Easier to compute than other measures (up to 12 qubits on a laptop)

PROPERTIES

- Faithfulness $\mathcal{G}(|\psi_S\rangle) = 0$
- Invariance under Clifford unitaries $\mathcal{G}(\hat{U}_C\,|\psi\rangle) = \mathcal{G}(|\psi\rangle)$
- Invariance under composition $\mathcal{G}(|\psi\rangle \otimes |\phi_S\rangle) = \mathcal{G}(|\psi\rangle)$
- Additivity $\mathcal{G}(|\psi\rangle_A \otimes |\phi\rangle_B) = \mathcal{G}(|\psi\rangle) + \mathcal{G}(|\phi\rangle)$

BOUNDS ON OPERATOR COST

Upper Bound

Given by existing algorithm

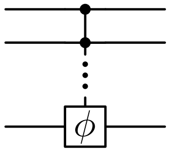
Lower Bound

No better algorithm is possible

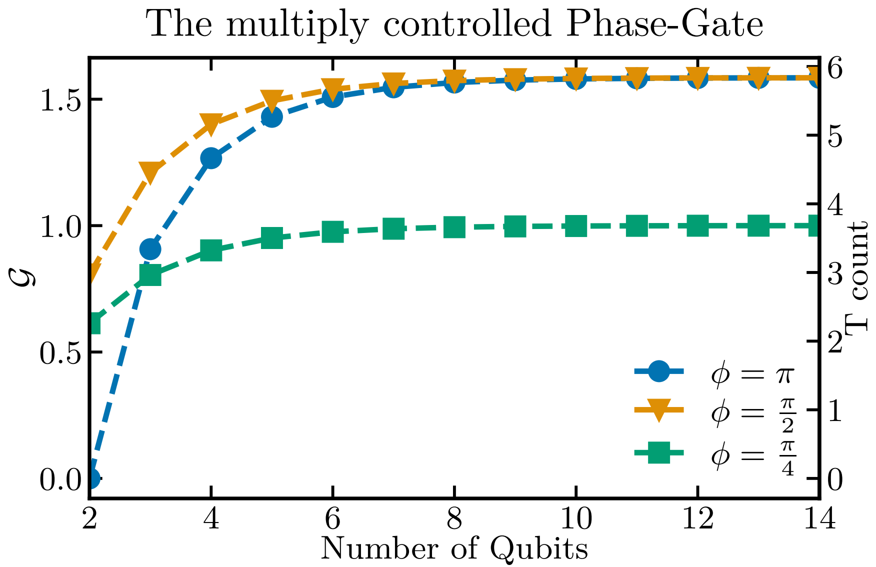
- Allows for optimization
- Use teleportation circuits and magic measures to find lower bounds
- Lower bound gate synthesis
  - e.g. how many T-gate needed at least to implement
- Form of our measures allows for analytical expressions

MULTIPLY CONTROLLED PHASE GATE

- Analytical solution for



$$M_\phi = \mathrm{diag}(1, \ldots, 1, e^{i\phi})$$



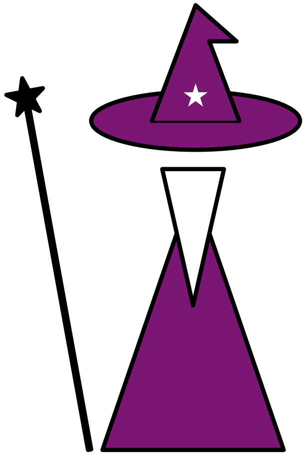
$$\phi = \pi : C^n Z$$

$$\phi = \frac{\pi}{2} : C^n S$$

$$\phi = \frac{\pi}{4} : C^n T$$

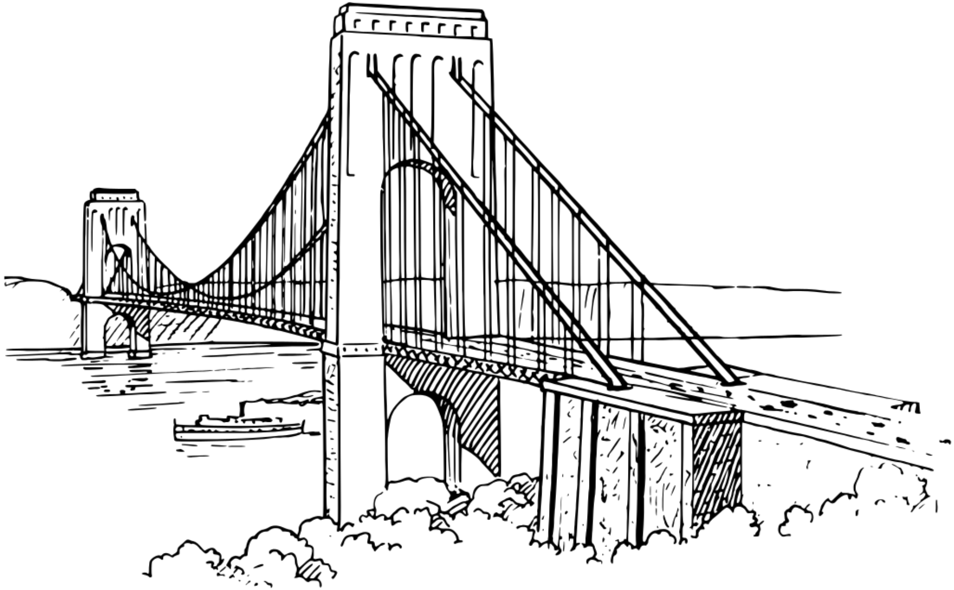
CONCLUSION

- Defined a magic measure
- Based on CV techniques and resources, we recover the st-norm and upgrade its status
- Its structure allows easier numerical computation and analytical values



OUTLOOK

- Generalise to qudits
- Transfer more results from CV to DV via error correction codes



CHALMERS

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BOUNDS

- Coincide with the ones obtain by the robustness

U_{target}	Robustness of Magic	GKP Magic	T-count
T_1	1.41421	0.272	1
$T_{1,2}$	1.74755	0.543	2
CS_{12}	2.2	0.807	3(2.967)
$T_{1,2,3}$	2.21895	0.815	3(2.996)
$CS_{12,13}$	2.55556	0.907	4(0.907)
T_1CS_{23}	2.80061	1.079	4(3.966)

MIXED QUBIT STATES

- The Wigner log. Negativity can be smaller than the one for pure stabilizer states
- Need attribute for that

$$\tilde{\mathcal{G}}(\hat{\rho}) = \max\left[0, \log_2\left(\frac{1}{\sqrt{\pi}^n}\sum_{\boldsymbol{i},\boldsymbol{j}\in\mathcal{F}_2^n}\left|\sum_{\boldsymbol{k}\in\mathcal{F}_2^n}(-1)^{\boldsymbol{i}\cdot\boldsymbol{k}}\rho_{\boldsymbol{k},\boldsymbol{k}+\boldsymbol{j}}\right|\right) - \log_2\left(\mathcal{N}_0(n)\right)\right]$$

- But for $\hat{\rho} = \hat{\rho}_I \otimes \hat{\rho}_O$

$$\tilde{\mathcal{G}}(\hat{\rho}_I \otimes \hat{\rho}_O) = \max\left[0, \log_2\left(\int_{\mathcal{C}_1} \mathrm{d}^n\boldsymbol{r}_1|W_{\hat{\rho}_I}|\int_{\mathcal{C}_2} \mathrm{d}^n\boldsymbol{r}_2|W_{\hat{\rho}_O}|\right) - \log_2\left(\mathcal{N}_0(n)\right)\right] = 0$$

CHOI-JAMIOLKOWSK ISOMORPHISM

- Bijection between channel and state

$$\hat{\varphi} = (\hat{\Phi} \otimes 1) |\psi\rangle \langle\psi| = \frac{1}{2} \sum_{j,k=0}^{1} \Phi(|j\rangle \langle k|) \otimes |j\rangle \langle k|$$

$$|\varphi_U\rangle = (\hat{U} \otimes 1) \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{j} \in \mathcal{F}_2^n} |\boldsymbol{j}, \boldsymbol{j}\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{1} |i, i\rangle$$