# Role of Quantum Coherence in Thermodynamics

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Quantum Resources:
From mathematical foundations to operational characterization
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# Time-Translation Symmetry

A quantum state $\rho \in \mathfrak{D}(A)$ is time-translation invariant if

$$e^{-iH^A t}\rho^A e^{iH^A t} = \rho^A \qquad \forall\, t \in \mathbb{R}$$

A quantum channel $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ is said to be time-translation covariant if

$$\mathcal{N}^{A\to B}\left(e^{-iH^A t}\rho^A e^{iH^A t}\right) = e^{-iH^B t}\mathcal{N}^{A\to B}\left(\rho^A\right) e^{iH^B t} \qquad \forall\, t \in \mathbb{R} \quad \forall\, \rho \in \mathfrak{D}(A)$$

$$\begin{tikzcd}
\rho^A \arrow[r, "\mathcal{N}"] \arrow[d, "\text{Time Evolution}"'] & \mathcal{N}(\rho^A) \arrow[d, "\text{Time Evolution}"] \\
e^{-iH^A t}\rho^A e^{iH^A t} \arrow[r, "\mathcal{N}"] & \mathcal{N}^{A\to B}\left(e^{-iH^A t}\rho^A e^{iH^A t}\right) = e^{-iH^B t}\mathcal{N}^{A\to B}\left(\rho^A\right) e^{iH^B t}
\end{tikzcd}$$

# Time-Translation Symmetry

## The Pinching Channel

The pinching channel is defined with respect to an Hamiltonian $H^A = \sum_{x=1}^{m} a_x \Pi_x^A$

$$\mathcal{P}^{A\to A}\left(\rho^A\right) := \sum_{x=1}^{m} \Pi_x^A \rho^A \Pi_x^A$$

**Properties:**

1. The pinching channel is itself time-translation covariant.
2. The pinching channel is idempotent; i.e. $\mathcal{P} = \mathcal{P} \circ \mathcal{P}$.
3. A density matrix $\rho \in \mathfrak{D}(A)$ is time-translation invariant iff $\mathcal{P}(\rho) = \rho$.
4. If $\mathcal{N} \in \mathrm{CPTP}(A \to B)$ is time-translation covariant then

$$\mathcal{P}^{B\to B} \circ \mathcal{N}^{A\to B} = \mathcal{N}^{A\to B} \circ \mathcal{P}^{A\to A}$$

5. The pinching channel can be expressed as a random unitary channel.

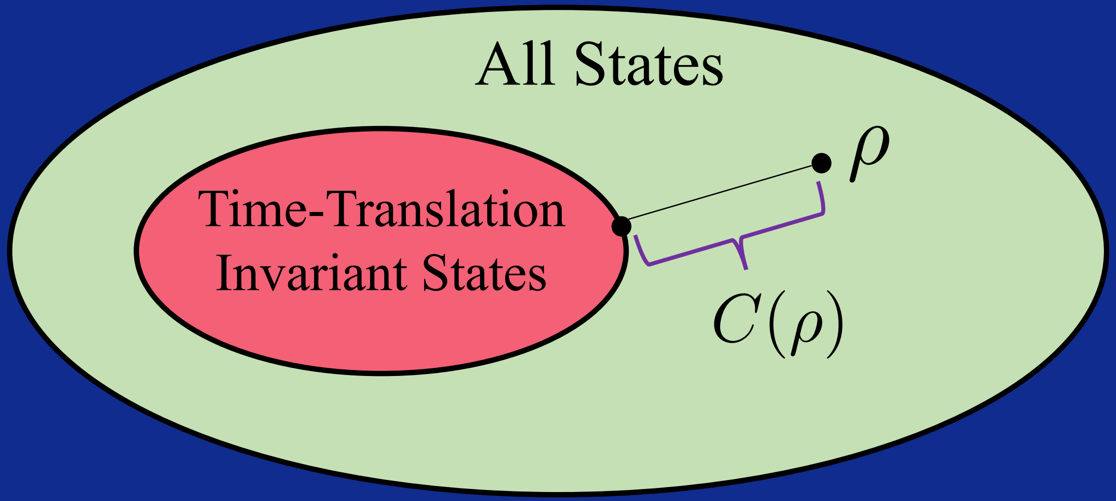
# Quantification of Time-Translation Asymmetry

Let $D(\rho\|\sigma) := \mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\rho\log\sigma]$ and $H(\rho) := -\mathrm{Tr}[\rho\log\rho]$. Then,

$$C(\rho) := \min_{\sigma\in\mathrm{INV}(A)} D(\rho\|\sigma) = H\left(\mathcal{P}(\rho)\right) - H(\rho)$$

For any $\rho \in \mathfrak{D}(A)$ the coherence of $n$ copies of $\rho$ satisfies

$$C\left(\rho^{\otimes n}\right) \leq |A|\log(n+1) \quad \xrightarrow{\text{Regularization}} \quad \lim_{n\to\infty}\frac{1}{n}C\left(\rho^{\otimes n}\right) = 0$$



# Manipulation of Time-Translation Asymmetry

Consider two Hamiltonians:

$$H^A = \sum_{x=1}^{m} a_x |x\rangle\langle x|^A \qquad\qquad H^B = \sum_{y=1}^{n} b_x |y\rangle\langle y|^B$$

Relatively non-degenerate Hamiltonians:

$$a_x - a_{x'} = b_y - b_{y'} \quad \Rightarrow \quad x = x' \;\text{ and }\; y = y'$$

**Theorem:** For relatively non-degenerate Hamiltonians

$$\mathcal{N} \in \mathrm{COV}(A \to B) \quad \iff \quad \mathcal{N} \text{ is classical; i.e. } \mathcal{N}^{A\to B} = \mathcal{P}^{B\to B} \circ \mathcal{N}^{A\to B} \circ \mathcal{P}^{A\to A}$$

# Manipulation of Time-Translation Asymmetry

Consider the Hamiltonian $H^A = \sum_{x=1}^{m} a_x |x\rangle\langle x|^A$

Non-degenerate Bohr Spectrum: $a_x - a_y = a_{x'} - a_{y'} \iff x = x' \text{ and } y = y'$

or $x = y \text{ and } x' = y'$

**Theorem:**

Let $A$ be a physical system with Bohr energy spectrum, and let $\rho, \sigma \in \mathfrak{D}(A)$. Then, the following are equivalent:

1. There exists a $\mathcal{N} \in \mathrm{COV}(A \to A)$ such that $\sigma = \mathcal{N}(\rho)$.
2. The matrix $Q = (q_{xy})$, whose components are given by

$$q_{xy} := \begin{cases} \min\left\{1, \frac{s_{xx}}{r_{xx}}\right\} & \text{if } x = y \\ \frac{s_{xy}}{r_{xy}} & \text{otherwise.} \end{cases}$$

is positive semidefinite.

$$r_{xy} := \langle x|\rho|y\rangle \neq 0$$

$$s_{xy} := \langle x|\sigma|y\rangle$$

# Manipulation of Time-Translation Asymmetry

**Corollary:**

Let $\sigma \in \mathfrak{D}(A)$ be an arbitrary state, and denote by $p_x := \langle x|\sigma|x\rangle$. Then,

$$|\psi\rangle := \sum_{x=1}^{m} \sqrt{p_x}|x\rangle$$

can be converted to $\sigma$ by a time-translation covariant channel.

# Quantum Thermodynamics

**Free States:** The Gibbs States $\gamma^A = \frac{e^{-\beta H^A}}{\mathrm{Tr}\left[e^{-\beta H^A}\right]}$ $\gamma^B = \frac{e^{-\beta H^B}}{\mathrm{Tr}\left[e^{-\beta H^B}\right]}$

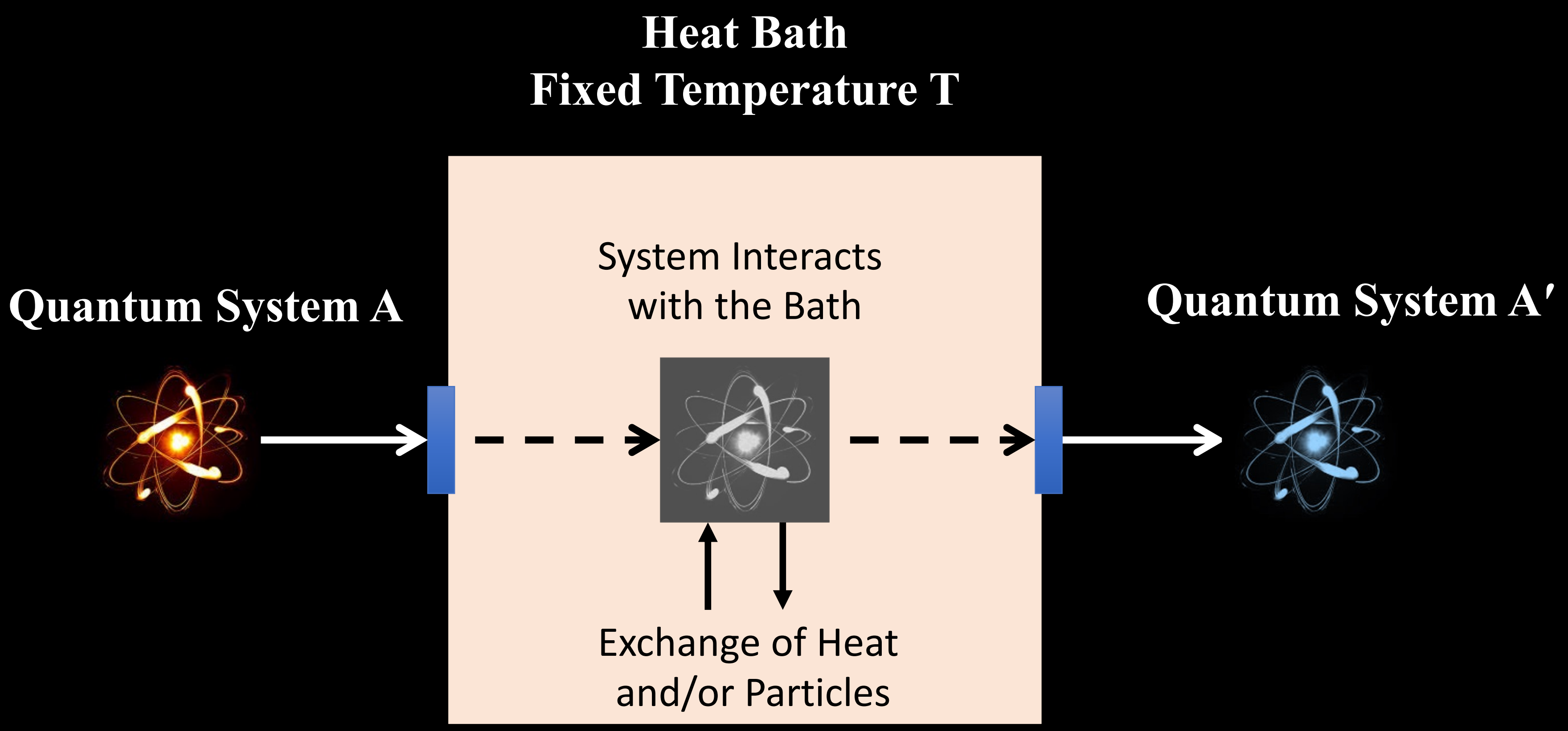
**Properties:**

1. Gibbs states cannot be used to extract work.

2. Gibbs states have minimal energy for a given entropy.

3. Gibbs states are the only completely passive states.

# Thermal Operations



# Thermal Operations

Three Basic Steps:

1. Thermal equilibrium. Any subsystem $B$, with Hamiltonian $H^B$, can be prepared in its thermal Gibbs state $\gamma^B$.

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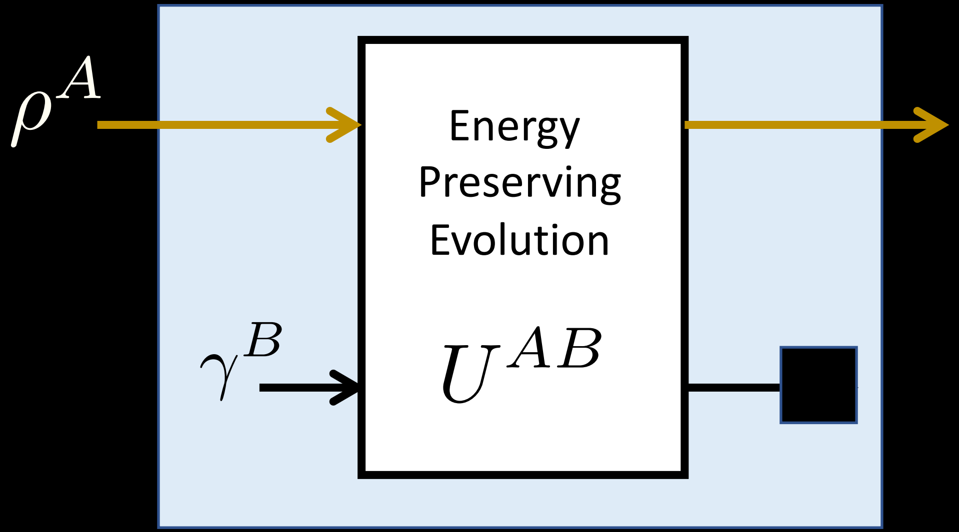
# Thermal Operations

Three Basic Steps:

1. Thermal equilibrium. Any subsystem $B$, with Hamiltonian $H^B$, can be prepared in its thermal Gibbs state $\gamma^B$.

2. Conservation of energy. Unitary operation on a composite physical system that commutes with the total Hamiltonian can be implemented.

3. Discarding subsystems. It is possible to trace over any subsystem (with a well defined Hamiltonian) of a composite system.




$$\mathcal{N}^{A\to A}\left(\rho^A\right) = \mathrm{Tr}_B\left[U^{AB}\left(\rho^A\otimes\gamma^B\right)U^{*AB}\right]$$

$$\mathcal{U}^{AB\to AB}\left(\gamma^{AB}\right) := U^{AB}\gamma^{AB}U^{*AB} = \gamma^{AB}$$

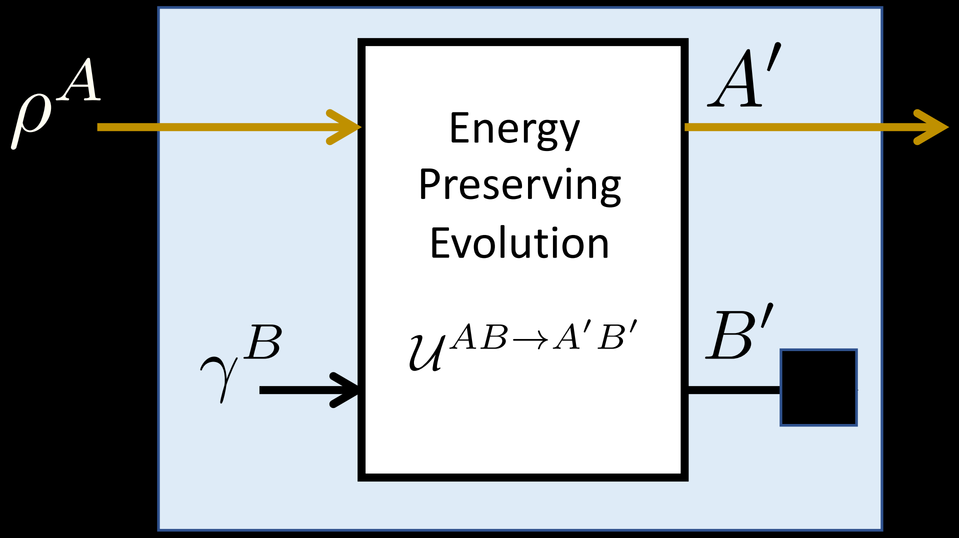
$$\gamma^{AB} := \gamma^A\otimes\gamma^B$$

# Thermal Operations

Consider a Gibbs preserving unitary channel $\mathcal{U} \in \mathrm{CPTP}(AB \to A'B')$

$$\mathcal{U}^{AB\to A'B'}\left(\gamma^A \otimes \gamma^B\right) = \gamma^{A'} \otimes \gamma^{B'} \qquad (\text{with } |A'B'| = |AB|)$$

**Lemma:** The quantum channel below is a thermal operation.




$$\mathcal{N}^{A\to A'}\left(\rho^A\right) = \mathrm{Tr}_{B'}\left[\mathcal{U}^{AB\to A'B'}\left(\rho^A \otimes \gamma^B\right)\right]$$

# Closed Thermal Operations

$\mathrm{TO}(A \to A')$ denotes the set of all thermal operations in $\mathrm{CPTP}(A \to A')$.

**Lemma:** The pinching channel $\mathcal{P} \in \mathrm{TO}(A \to A)$.

Two undesirable properties:

1. The set $\mathrm{TO}(A \to A')$ is not closed in $\mathrm{CPTP}(A \to A')$.
2. The set $\mathrm{TO}(A \to A')$ is not convex.

The closure of the set $\mathrm{TO}(A \to A')$:

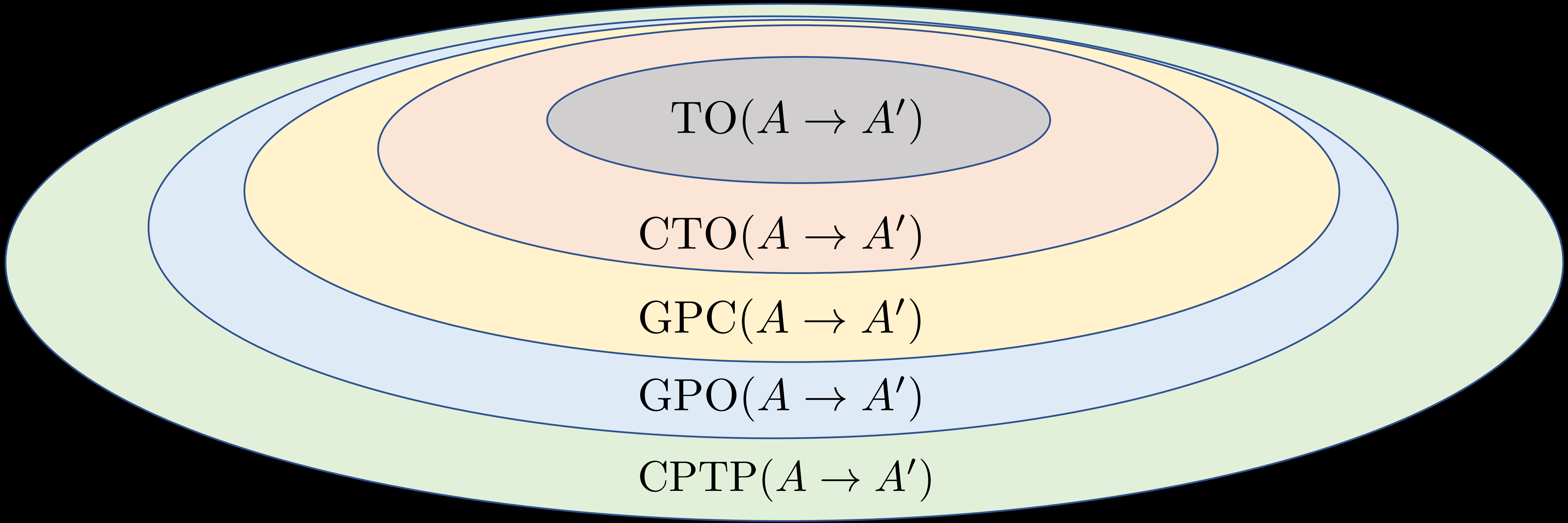
$$\mathrm{CTO}(A \to A') := \left\{ \mathcal{N} \in \mathrm{CPTP}(A \to A') \ : \ \mathcal{N} = \lim_{k \to \infty} \mathcal{N}_k \ , \ \mathcal{N}_k \in \mathrm{TO}(A \to A') \right\}$$

**Theorem:** The set $\mathrm{CTO}(A \to A')$ is closed and convex.

# Gibbs Preserving Covariant Operations

Every thermal operation $\mathcal{N} \in \mathrm{CPTP}(A \to A')$ has two key properties:

1. $\mathcal{N}^{A\to A'}$ is *Gibbs preserving operation* (GPO); that it, $\mathcal{N}^{A\to A'}(\gamma^A) = \gamma^{A'}$.
2. $\mathcal{N}^{A\to A'}$ is time-translation covariant; i.e. $\mathcal{N} \in \mathrm{COV}(A \to A')$.



# Quantification of Quantum Athermality

**The Relative Entropy of a Resource:**

$$D\left(\rho \| \gamma\right) = -H(\rho) - \mathrm{Tr}\left[\rho \log \gamma\right]$$

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&= -H(\rho) - \mathrm{Tr}\left[\mathcal{P}(\rho) \log \gamma\right] \\
&= D\left(\mathcal{P}(\rho)\big\|\gamma\right) + H\big(\mathcal{P}(\rho)\big) - H(\rho)
\end{aligned}
$$

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&= -H(\rho) - \mathrm{Tr}\left[\mathcal{P}(\rho) \log \gamma\right] \\
&= D\big(\mathcal{P}(\rho)\big\|\gamma\big) + H\big(\mathcal{P}(\rho)\big) - H(\rho) \\
&= D\big(\mathcal{P}(\rho)\big\|\gamma\big) + C(\rho)
\end{aligned}
$$

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Nonuniformity

Quantum Coherence

# Deterministic Manipulation of Quantum Athermality

Suppose $\mathcal{P}(\rho) = \mathcal{P}(\sigma)$

$$(\rho, \gamma) \xrightarrow{\text{GPC}} (\sigma, \gamma)$$

$$\rho \xrightarrow[\phantom{x}]{\substack{\text{Time-Translation} \\ \text{Covarinat Operations}}} \sigma$$

$$Q^A := I^A + \sum_{x \neq y \in [m]} \frac{s_{xy}}{r_{xy}} |x\rangle\langle y|^A \geq 0$$

# The Quasi-Classical Case

$$[\rho^A, \gamma^A] = 0 \quad \Rightarrow \quad \rho^A \sim \mathbf{p}^A \quad , \quad \gamma^A \sim \mathbf{g}^A$$

**Theorem:**

$$(\mathbf{p}^A, \mathbf{g}^A) \xrightarrow{\text{CTO}} (\mathbf{q}^B, \mathbf{g}^B) \quad \Longleftrightarrow \quad (\mathbf{p}^A, \mathbf{g}^A) \xrightarrow{\text{GPO}} (\mathbf{q}^B, \mathbf{g}^B)$$

D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, International Journal of Theoretical Physics 39, 2717 (2000)

**Relative Majorization:**

$$(\mathbf{p}^A, \mathbf{g}^A) \xrightarrow{\text{GPO}} (\mathbf{q}^B, \mathbf{g}^B) \quad \Longleftrightarrow \quad \mathbf{q}^B = E\mathbf{p}^A \quad \text{and} \quad \mathbf{g}^B = E\mathbf{g}^A$$

$$\Longleftrightarrow \quad (\mathbf{p}^A, \mathbf{g}^A) \succ (\mathbf{q}^B, \mathbf{g}^B)$$

# The Church of the Trivialized Hamiltonian

For a trivial Hamiltonian $H^A = 0$ the Gibbs states is uniform:

$$\mathbf{g}^A = \mathbf{u}^{(m)} := \frac{1}{m} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad m := |A|$$

**Theorem:**

If $\mathbf{g} = (\frac{k_1}{k}, \ldots, \frac{k_m}{k})^T$ has rational components, then for any probability vector $\mathbf{p} = (p_1, \ldots, p_m)^T$

$$(\mathbf{p}, \mathbf{g}) \xleftrightarrow{\text{CTO}} (\mathbf{r}, \mathbf{u}^{(k)}) \quad \text{where} \quad \mathbf{r} := \bigoplus_{x=1}^{m} p_x \mathbf{u}^{(k_x)}$$

# The Golden Unit of Athermality

$$(|0\rangle\langle 0|^A, \mathbf{u}^A)$$

Pure State Maximally Mixed State.

**Theorem:** $(|0\rangle\langle 0|^A, \mathbf{u}^A) \xleftrightarrow{\text{CTO}} (|0\rangle\langle 0|^X, \mathbf{u}_m^X)$

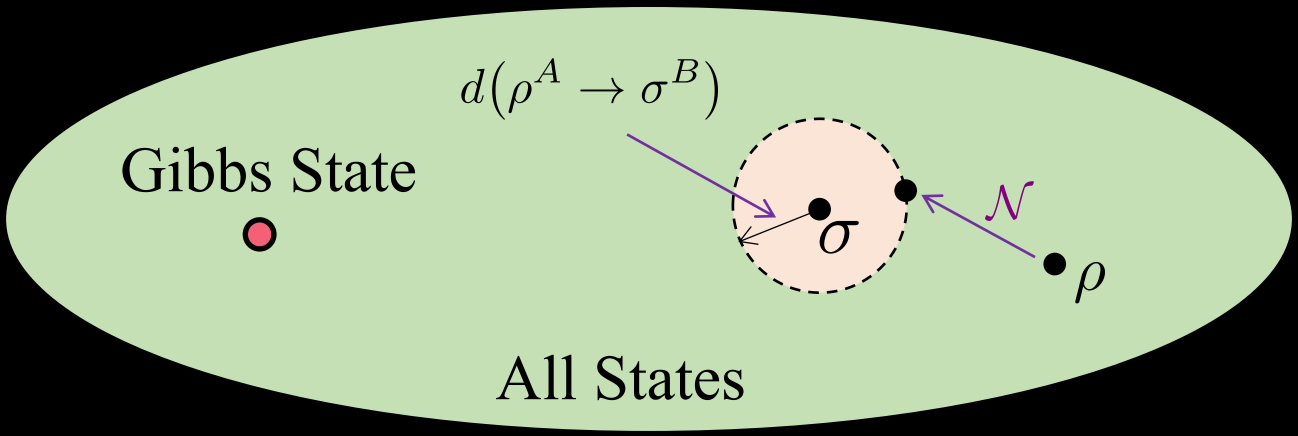
$|A| = m$ $|X| = 2$

$$\mathbf{u}_m^X := \frac{1}{m}|0\rangle\langle 0|^X + \frac{m-1}{m}|1\rangle\langle 1|^X .$$

Xin Wang & Mark Wilde, Phys. Rev. Research 1, 033170 (2019)

# Distillation

$$d(\rho^A \to \sigma^B) := \min_{\mathcal{N} \in \mathrm{CTO}(A \to B)} \frac{1}{2} \left\| \sigma^B - \mathcal{N}\left(\rho^A\right) \right\|_1$$




$$\mathrm{Distill}^{\varepsilon}(\rho) = \log \sup_{0<m\in\mathbb{R}} \left\{ m \ : \ d\Big( (\rho^A, \gamma^A) \to \left( |0\rangle\langle 0|^X, \mathbf{u}_m^X \right) \Big) \le \varepsilon \right\}$$

$$= D_{\min}^{\varepsilon}\left( \mathcal{P}(\rho) \middle\| \gamma \right)$$

**Hypothesis Testing Divergence:** $D_{\min}^{\varepsilon}(\rho\|\gamma) := \min_{0 \le \Lambda \le I^A} \left\{ \mathrm{Tr}[\gamma\Lambda] \ : \ \mathrm{Tr}[\Lambda\rho] \ge 1 - \varepsilon \right\}$

# Distillation

$$\mathrm{Distill}^{\varepsilon}(\rho) = \log \sup_{0<m\in\mathbb{R}} \left\{ m \;:\; d\Big( (\rho^A, \gamma^A) \to \big( |0\rangle\langle 0|^X, \mathbf{u}_m^X \big) \Big) \le \varepsilon \right\}$$

$$= D_{\min}^{\varepsilon}\left( \mathcal{P}(\rho) \| \gamma \right)$$

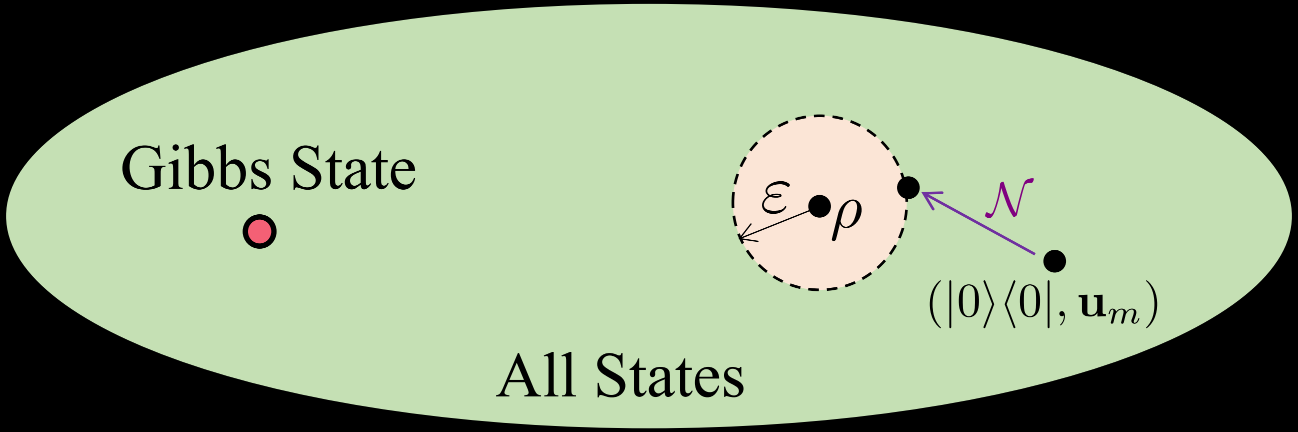
**Asymptotically:**

$$\mathrm{Distill}(\rho) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \mathrm{Distill}^{\varepsilon}\left( \rho^{\otimes n} \right)$$

$$= \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} D_{\min}^{\varepsilon}\left( \mathcal{P}(\rho^{\otimes n}) \| \gamma \right)$$

$$= D(\rho\|\gamma)$$

# Cost




$$\mathrm{Cost}^{\varepsilon}(\rho) = \log \inf_{0<m\in\mathbb{R}} \left\{ m \; : \; d\Big( (|0\rangle\langle 0|^X, \mathbf{u}_m^X) \to (\rho^A, \gamma^A) \Big) \leq \varepsilon \right\}$$

$= D_{\max}^{\varepsilon}(\rho\|\gamma)$ (Under Gibbs Preserving Operations)

$= \infty$ (Under GPC or CTO or TO)

Xin Wang & Mark Wilde, Phys. Rev. Research 1, 033170 (2019)

# Sublinear Athermality Resources (SLAR)

**Definition:** A sublinear athermality resource (SLAR) is a sequence of quantum athermality systems $\{R_n\}_{n\in\mathbb{N}}$, such that:

1. $|R_n| = \mathrm{Poly}(n)$
2. There exists $c > 0$ and $0 \leq \alpha < 1$ such that $\left\|H^{R_n}\right\|_\infty \leq cn^\alpha$ for all $n \in \mathbb{N}$.

$$\left(|0\rangle\langle 0|, \mathbf{u}_m\right) \xrightarrow{\text{CTO+SLAR}} \left(\rho^{\otimes n}, \gamma^{\otimes n}\right)$$

$$\Updownarrow$$

$$\left(|0\rangle\langle 0|, \mathbf{u}_m\right) \otimes \left(\eta^{R_n}, \gamma^{R_n}\right) \xrightarrow{\text{CTO}} \left(\rho^{\otimes n}, \gamma^{\otimes n}\right)$$

# Asymptotic Cost

$$(|0\rangle\langle 0|, \mathbf{u}_m) \xrightarrow{\text{CTO+SLAR}} (\rho^{\otimes n}, \gamma^{\otimes n})$$

$$\Updownarrow$$

$$(|0\rangle\langle 0|, \mathbf{u}_m) \otimes \left(\eta^{R_n}, \gamma^{R_n}\right) \xrightarrow{\text{CTO}} (\rho^{\otimes n}, \gamma^{\otimes n})$$

**Theorem:** If $\rho$ is a pure state then

$$\mathrm{Cost}(\rho) := \lim_{n\to\infty} \frac{m}{n} = D(\rho\|\gamma)$$

# Future Work

State with zero non-uniformity and the most coherence for a given Gibbs state:

$$|\psi_\gamma\rangle := \sum_{x=1}^{m} \sqrt{g_x}|x\rangle \quad \text{and} \quad \gamma = \sum_{x=1}^{m} g_x |x\rangle\langle x|$$

Interconversions with two types of resources:

$$\big(\psi_\gamma, \gamma\big)^{\otimes k} \otimes \big(|0\rangle\langle 0|, \mathbf{u}\big)^{\otimes m} \xrightarrow{\text{GPC}} \big(\rho, \gamma\big)^{\otimes n}$$

Coherence/Asymmetry

Nonuniformity

Thank You!