A hierarchy of resource theories of quantum incompatibility

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introduction

In quantum theory, some measurements necessarily exclude others.

This is what enables quantum algorithms, QKD protocols, violations of Bell's inequalities, etc.

Various formalizations: preparation URs, measurement (noise–disturbance) URs, and incompatibility.

compatible POVMs

Definition

Given a family $(P_x^{(i)})_{x\in\mathsf{X}, i\in\mathsf{I}}$ of POVMs, all defined on the same system A, we say that the family is *compatible*, whenever there exists a mother POVM $(O_w)_{w\in\mathsf{W}}$ on system A and a family of conditional probability distributions $\mu(x|w,i)$ such that

$$P_x^{(i)} = \sum_w \mu(x|w,i) O_w \ ,$$

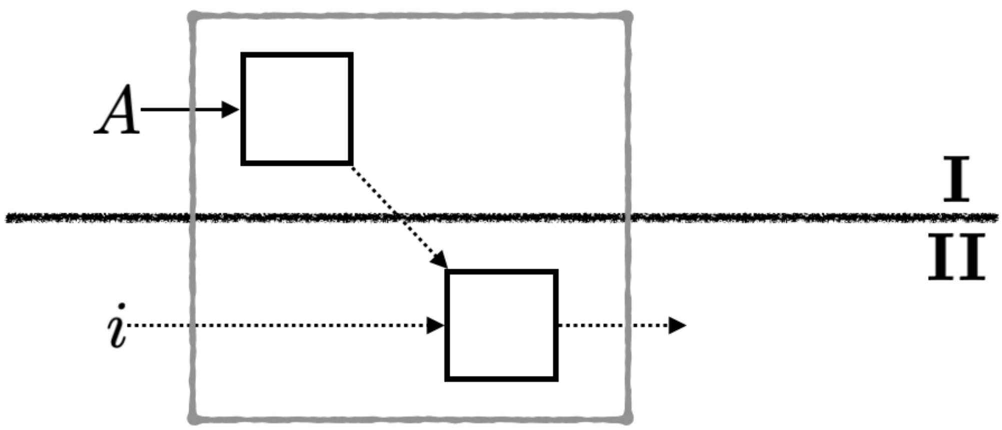
for all $x \in \mathsf{X}$ and all $i \in \mathsf{I}$.

families of POVMs as one "programmable" POVM

Whenever we have a family of objects (states, channels, POVMs, etc) it can be useful to see it as a single programmable device.

In what follows, we will characterize (in)compatibility in terms of a hierarchy of constraints on how the system and the program, seen as two separate parties, can "communicate".

compatible programmable POVMs



See [F.B., E. Chitambar, W. Zhou; PRL 2020].

from POVMs to instruments

many different incompatibilities

While for POVMs consensus exists for a unique notion of compatibility, in the case of instruments the situation is not so clear.

classical compatibility 1/2

Definition

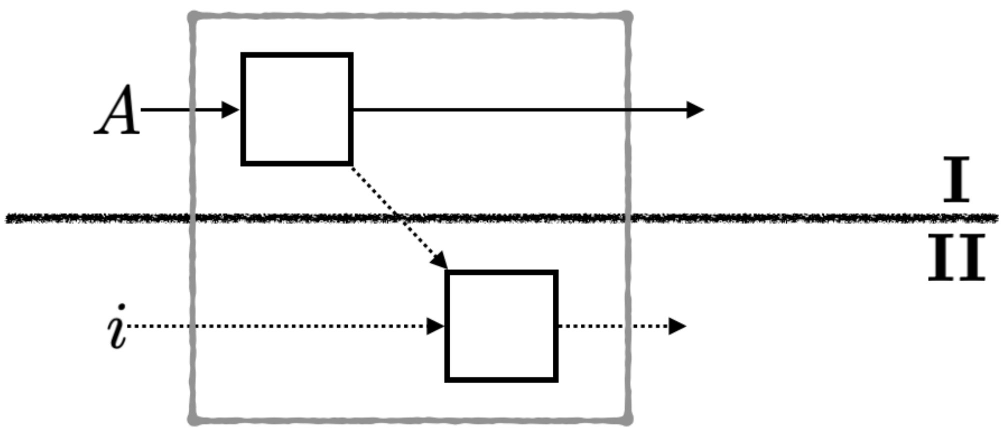
Given a family of instruments $(\mathcal{I}_x^{(i)})_{x\in\mathsf{X},i\in\mathsf{I}}$, all defined on the same system A, we say that the family is *classically compatible*, whenever there exists a mother instrument $(\mathcal{H}_w)_{w\in\mathsf{W}}$ on A and a family of conditional probability distributions $\mu(x|w,i)$ such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)\mathcal{H}_w ,$$

for all $x \in \mathsf{X}$ and all $i \in \mathsf{I}$.

We call this "classical" because it involves only classical post-processings, but it is also called "traditional" [Mitra and Farkas; PRA, 2022].

classical compatibility 2/2



Crucially:

- no shared entanglement and communication is classical
- communication goes only from $\mathbf{I}$ to $\mathbf{II}$, i.e., the above is necessarily $\mathbf{II}\to\mathbf{I}$ non-signaling, see [Ji and Chitambar; PRA (2021)]

parallel compatibility 1/2

Without loss of generality (classical labels can be copied), compatible POVMs may be assumed to be recovered by marginalization, i.e.,

$$P_x^{(i)} = \sum_{x_j : j \neq i} O_{x_1, x_2, \ldots, x_n}$$

The notion of "parallel compatibility" for instruments lifts the above insight to the quantum outputs.

parallel compatibility 2/2

Definition (Heinosaari–Miyadera–Ziman, 2015)

Given a family of instruments $(\mathcal{I}_x^{(i)})_{x \in \mathsf{X}, i \in \mathsf{I}}$, all acting on the same system A but with possibly different output systems B_i, we say that the family is *parallelly compatible*, whenever there exist

- a mother instrument $(\mathcal{H}_w)_{w \in \mathsf{W}}$ from A to $\otimes_{i \in \mathsf{I}} B_i$;
- and a family of conditional probability distributions $\mu(x|w,i)$,

such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i) [\mathrm{Tr}_{B_{i' : i' \neq i}} \circ \mathcal{H}_w] \ ,$$

for all $x \in \mathsf{X}$ and all $i \in \mathsf{I}$.

parallel compatibility VS classical compatibility

- parallel compatibility is able to go beyond no-signaling, hence, parallelly compatibile $\not\Longrightarrow$ classically compatibile
- however, parallel compatibility departs from the "no information without disturbance" tenet, because non-disturbing instruments are never parallelly compatible. Example:
  - take $(\mathcal{I}_x)_x$ and $(\mathcal{J}_y)_y$, with $\mathcal{I}_x \propto \mathcal{J}_y \propto \mathrm{id}$, i.e., both instruments do not touch the quantum system and output purely random outcomes
  - these two instruments are obviously classically compatible; however, they *cannot* be parallelly compatible, otherwise we would violate the no-broadcasting theorem
- hence classically compatibile $\not\Longrightarrow$ parallelly compatibile

Closing the gap

q-compatibility 1/2

Definition

Given a family of instruments $(\mathcal{I}_x^{(i)})_{x \in \mathsf{X}, i \in \mathsf{I}}$, all acting on the same system A but with possibly different output systems B_i, we say that the family is *q-compatible*, whenever there exist

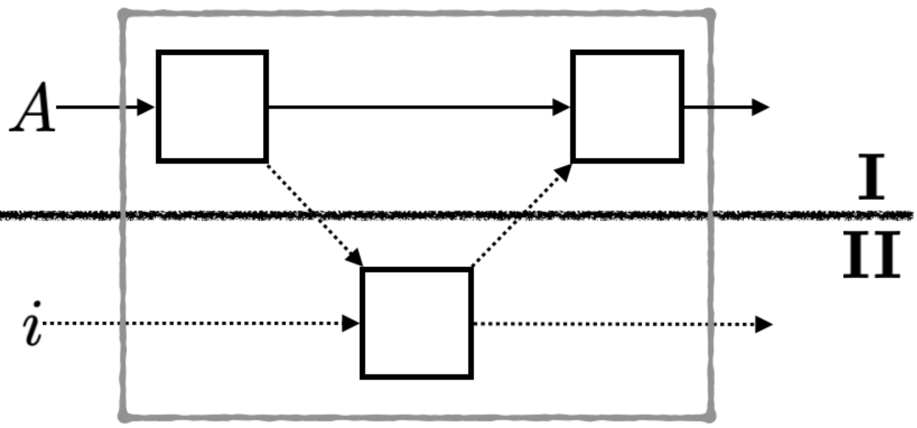
- a mother instrument $(\mathcal{H}_w)_{w \in \mathsf{W}}$ from A to C;
- a family of conditional probability distributions $\mu(x|w,i)$;
- and a family of channels $(\mathcal{D}^{(x,w,i)} : C \to B_i)_{x \in \mathsf{X}, w \in \mathsf{W}, i \in \mathsf{I}}$

such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)[\mathcal{D}^{(x,w,i)} \circ \mathcal{H}_w] \ ,$$

for all $x \in \mathsf{X}$ and all $i \in \mathsf{I}$.

q-compatibility 2/2

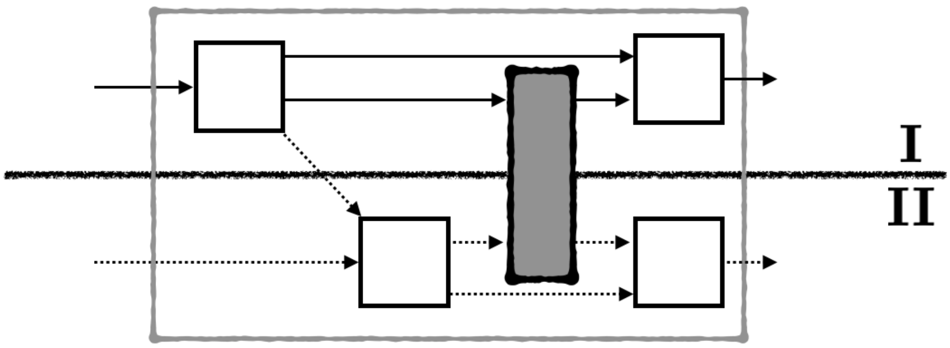


Crucially:

- no shared entanglement and communication is classical
- only one interactive round **I**$\to$ **II**$\to$**I**
- both classical and parallel compatibilities are special cases of q-compatibility

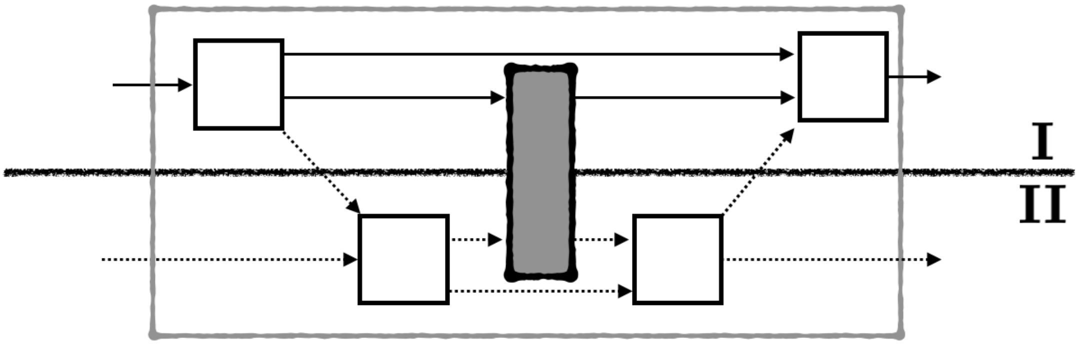
a strict hierarchy of resource theories of instruments incompatibility

classical incompatibility: free operations (T_{cl})



- all cassically compatible devices can be created for free
- if the initial device (the dark gray inner box) is classically compatible, the final device is also classically compatible

q-incompatibility: free operations (T_q)



- all q-compatible devices can be created for free
- if the initial device (the dark gray inner box) is q-compatible, the final device is also q-compatible

construction of the resource monotones

classical-quantum guessing games 1/2

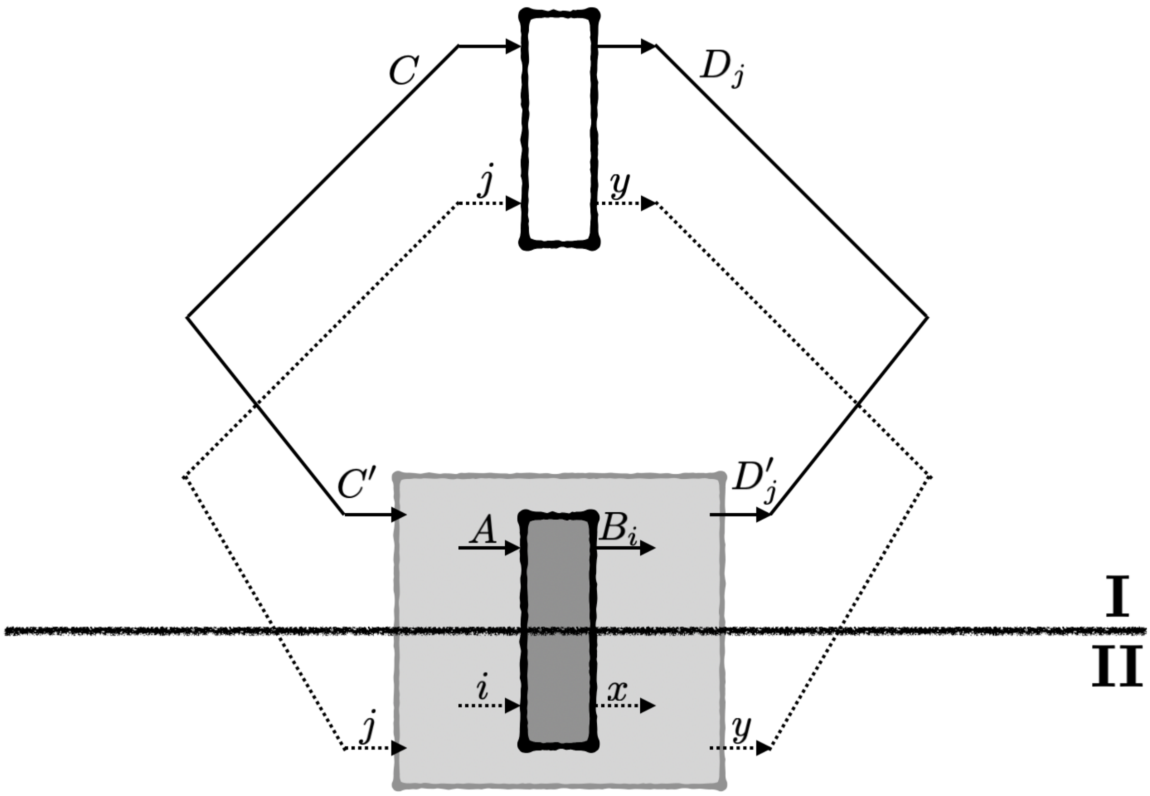
Definition

Two spatially separated players, $\mathbf{I}$ and $\mathbf{II}$, initially share a programmable instrument $(\mathcal{I}_x^{(i)} : A \to B_i)_{x\in\mathsf{X},i\in\mathsf{I}}$. A referee chooses a reference programmable instrument $(\mathcal{K}_y^{(j)} : C \to D_j)_{y\in\mathsf{Y},j\in\mathsf{J}}$. In each round, the referee picks a program value at random from the set J and sends it to $\mathbf{II}$. At the same time, the referee prepares a maximally entangled state $\Phi^+_{CC'}$ and sends the C' system to $\mathbf{I}$. For each operational framework, T_{cl} or T_q, the expected utility associated to $(\mathcal{I}_x^{(i)})_{x,i}$ is computed as

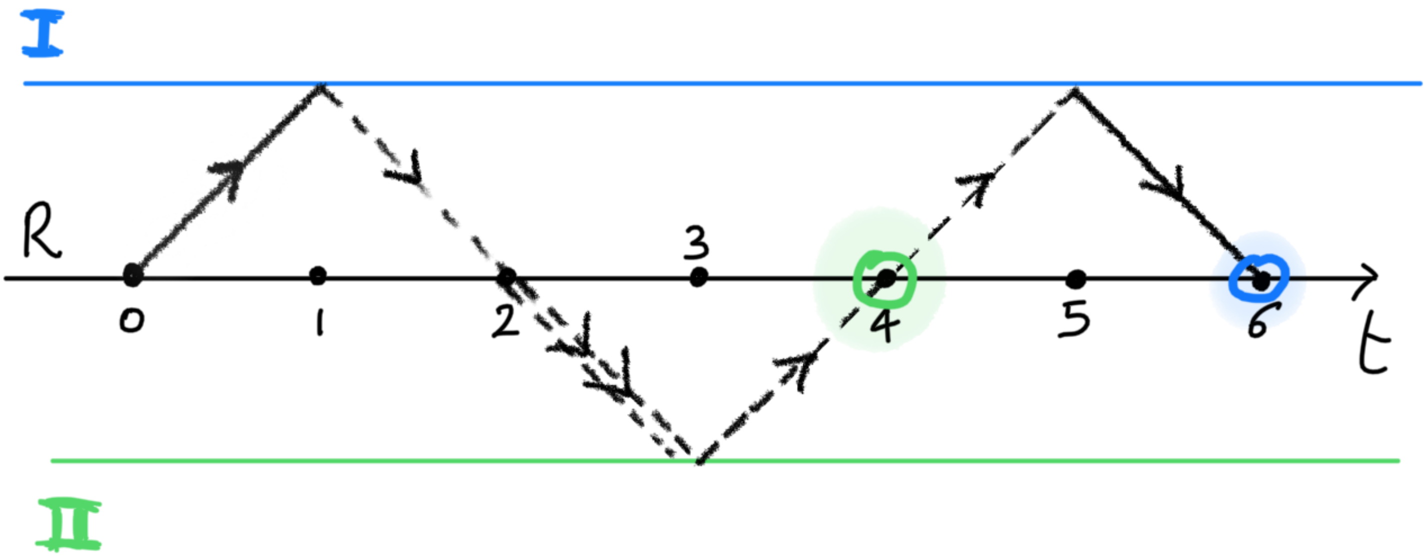
$$u_\bullet((\mathcal{I}_x^{(i)});(\mathcal{K}_y^{(j)})) := \max_{\mathbb{T}\in\mathsf{T}_\bullet}\sum_{j,y}\langle\Phi^+_{D_jD'_j}|(\mathcal{K}_y^{(j)}\otimes[\mathbb{T}\mathcal{I}]_y^{(j)})(\Phi^+_{CC'})|\Phi^+_{D_jD'_j}\rangle\ ,$$

where $\bullet\in\{cl,q\}$.

classical-quantum guessing games 2/2



constraining communication by timing



incompatibility preorders

Definition

Given two programmable instruments $(\mathcal{I}_x^{(i)})_{x,i}$ and $(\mathcal{J}_y^{(j)})_{y,j}$, we write

$$(\mathcal{I}_x^{(i)})_{x,i} \supseteq_\bullet (\mathcal{J}_y^{(j)})_{y,j} \ , \qquad \bullet \in \{cl, q\} \ ,$$

whenever $u_\bullet((\mathcal{I}_x^{(i)}); (\mathcal{K}_y^{(j)})) \geqslant u_\bullet((\mathcal{J}_y^{(j)}); (\mathcal{K}_y^{(j)}))$, for all distributed classical-quantum guessing games $(\mathcal{K}_y^{(j)})_{y,j}$.
We also write

$$(\mathcal{I}_x^{(i)})_{x,i} \succeq_\bullet (\mathcal{J}_y^{(j)})_{y,j} \ , \qquad \bullet \in \{cl, q\} \ ,$$

whenever there exists a superoperation in $\mathsf{T}_\bullet$ that is able to transform $(\mathcal{I}_x^{(i)})_{x,i}$ into $(\mathcal{J}_y^{(j)})_{y,j}$.

Theorem

The equivalence relation holds, with $\bullet \in \{cl, q\}$*:*

$$(\mathcal{I}_x^{(i)})_{x,i} \supseteq_\bullet (\mathcal{J}_y^{(j)})_{y,j} \quad \Longleftrightarrow \quad (\mathcal{I}_x^{(i)})_{x,i} \succeq_\bullet (\mathcal{J}_y^{(j)})_{y,j}$$

Corollary

A programmable instrument is not $\bullet$*-compatible if and only if there exists a classical-quantum guessing game* $(\mathcal{K}_y^{(j)})_{y,j}$ *that is able to witness the separation, that is*

$$u_\bullet((\mathcal{I}_x^{(i)}); (\mathcal{K}_y^{(j)})) > u_\bullet^\star((\mathcal{K}_y^{(j)})) \ ,$$

where $u_\bullet^\star((\mathcal{K}_y^{(j)}))$ *is the maximum utility that can be obtained with* $\bullet$*-compatible devices.*

conclusions

- for instruments we had (at least) two inequivalent notions of compatibility (both recovering the unique notion of POVM compatibility in the case of instruments with trivial quantum output)
- q-compatibility unifies them within a hierarchy of (complete and operational) resource theories of bipartite communication
- we get a better picture of the relations between incompatibility, no-signaling, no-broadcasting, and the "no info w/o disturbance" principle in quantum theory
- not featured in this talk and/or work-in-progress: "compatibility" VS "no-exclusivity", higher-order operations, incompatibility witnesses and semiquantum tests, the case of GPTs

The End: Thank You!