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General entropic constraints on CSS codes for magic distillation protocols

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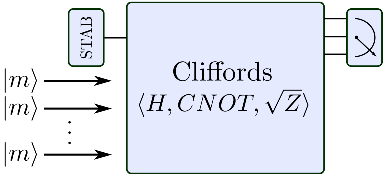
Outline

I. **Setting the scene**
   Magic distillation & majorization on discrete phase space (odd d)

II. **A (restricted) stat mech framework for qubit magic**
   Entropic conditions for generic completely CSS-preserving channels

III. **Application to protocols based on CSS codes**
   Upper and lower bounds on code parameters

Magic state injection model



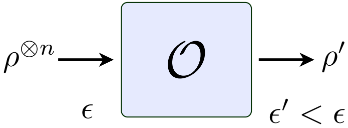
magic $\{|m\rangle\}$ + stabilizer operations $\mathcal{O}_{STAB}$ = universal quantum computing

Problem: magic states noisy

Solution: protocol to purify

Gottesman (1998)

Magic state distillation



- **Problem:** Existing protocols costly!
- Can we do better?
- Want constraints on resource cost/ overhead ($\approx n$)
- **Prior work:** *Lower bounds* on *n* from monotones[1] or majorization[2]

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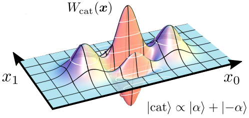
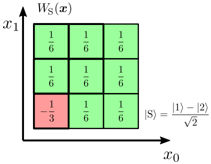
[1] Fang and Liu; Seddon et al.; Regula (2020; 2021; 2022)
[2] Koukoulekidis and Jennings (2022)

Majorization on discrete phase space

Discrete phase space





- state $\rho \Leftrightarrow$ quasi-distribution W_ρ
- channel $\mathcal{E} \Leftrightarrow$[3] matrix $W_{\mathcal{E}} = d^2 W_{J(\mathcal{E})}$:

$$W_{\mathcal{E}(\rho)}(\boldsymbol{y}) = \sum_{\boldsymbol{x}} W_{\mathcal{E}}(\boldsymbol{y}|\boldsymbol{x}) W_\rho(\boldsymbol{x})$$

Odd *d*:

ρ magic $\Rightarrow$ negativity in W_ρ[4]

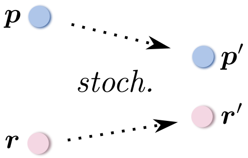
$\mathcal{E} \in \mathcal{O}_{SO} \Rightarrow W_{\mathcal{E}}$ stochastic

i.e., $W_{\mathcal{E}}(\boldsymbol{y}|\boldsymbol{x}) \geq 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{P}_d$

[3] Wang, Wilde, and Su 2019.
[4] Veitch, Ferrie, et al. 2012.

Relative majorization

- c.f. thermodynamics: stochasic processing $\sim$ *“free energy non-increasing”*[5]



$$\Longrightarrow \quad \Delta F_\alpha \sim \ \Delta D_\alpha \leq 0$$

$$\Updownarrow$$

$$(\boldsymbol{p}, \boldsymbol{r}) \succ (\boldsymbol{p}', \boldsymbol{r}')$$

$$\Delta D_\alpha := D_\alpha(\boldsymbol{p}'||\boldsymbol{r}') - D_\alpha(\boldsymbol{p}||\boldsymbol{r}) \leftarrow (\alpha\text{-Rényi divergences})$$

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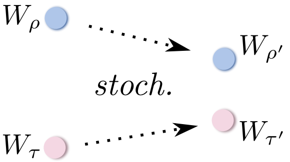
[5] Brandao et al. (2015)

Framework for MSQC in odd *d*

- Extends[5,6] to quasi-distributions for $\alpha \in \mathcal{A}$



Stochastic (stabilizer) processing of quasi-dristributions (magic)

relative to some distinguished reference process

But: most algorithms use qubits!

[5] Koukoulekidis and Jennings (2022)
[6] $\mathcal{A} := \{\frac{2a}{2b-1} : a \geq b \geq 1\}$

Q: *Can we extend to qubits?*

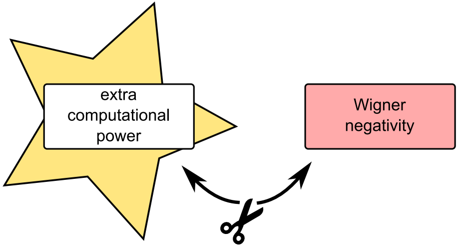
Problem with Qubits

- Known difficulties constructing W_ρ with "nice properties"

- There is no representation satisfying

$$W_{\mathcal{E}_2 \otimes \mathcal{E}_1} = W_{\mathcal{E}_2} \otimes W_{\mathcal{E}_1} \text{ and } W_{\mathcal{E}_2 \circ \mathcal{E}_1} = W_{\mathcal{E}_2} W_{\mathcal{E}_1},$$

  such that all $\mathcal{O}_{STAB}$ are stochastic[7]



[7] Schmid et al. (2022)

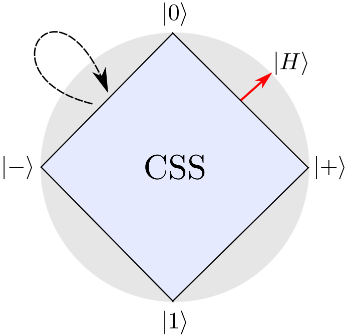
Approach

1) Restrict to distillation protocols based on[8]

$\mathcal{O}_{css}$:= *completely CSS-preserving operations*

2) Construct representation W_ρ:

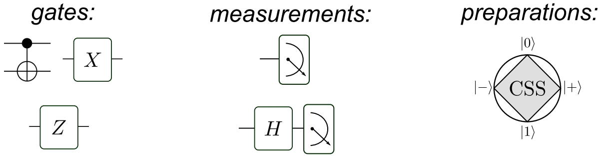
$\mathcal{E} \in \mathcal{O}_{css} \implies W_{\mathcal{E}}$ stochastic



Penalty: W_ρ complex-valued unless ρ rebit (e.g. $|H\rangle \overset{\mathcal{C}}{\leftrightarrow} T|+\rangle$)

[8]Delfosse et al. 2015; Catani and Browne 2018.

Completely CSS-preserving operations?



- $\mathcal{O}_{css}$ + *rebit magic states* = universal quantum computing[9]
- Many existing distillation protocols based on $\mathcal{O}_{css}$

e.g. CSS code projections
(seminal 15-1 protocol[10])

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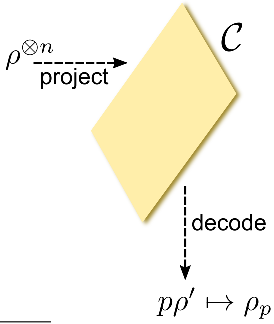
[9] Delfosse et al. (2015)
[10] Bravyi and Kitaev (2005)

CSS code projections

In practice: *project onto codespace of* $[[n, k, D]]$ *CSS code*[11] $\mathcal{C}$

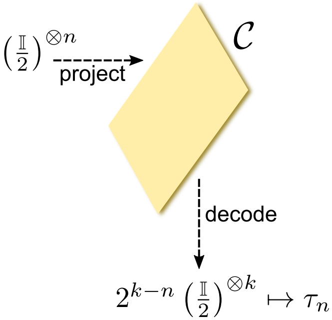
(X-type/ Z-type generators only)



[11] Bravyi and Kitaev (2005)
Campbell and Browne (2009)

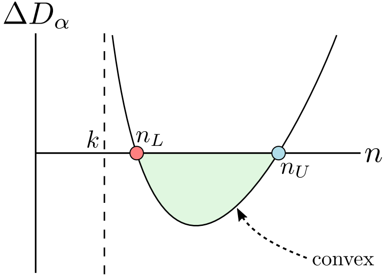
Choosing reference process

Intuition: Code projection protocols are *sub-unital* (since, $\Pi = \mathbb{I}_L$)



Entropic constraints

Result: *If an $[[n,k,D]]$ CSS code projection $\rho^{\otimes n} \mapsto p\rho'$ exists, then $\Delta D_\alpha \leq 0$, $\forall \alpha \in \mathcal{A}$.*



$\Rightarrow$ Lower (n_L) **and upper** (n_U) bounds on *n*

Can compute numerically using root-finding methods, or...

$$\Delta D_\alpha \coloneqq D_\alpha(W_{\rho_p}||W_{\tau_n}) - nD_\alpha(W_\rho||W_{\frac{\mathbb{1}}{2}})$$

Analytic upper and lower bounds

Theorem 1: Bounds on resource cost

Let ρ, ψ be rebit magic states. If $\rho^{\otimes n} \mapsto p\rho'$, where $\left\|\rho' - \psi^{\otimes k}\right\|_1 \leq \epsilon'$, under an $[[n, k, D]]$ CSS code projection, then for any $\alpha \in \mathcal{A}$

$$n \leq \frac{k\left[H_\alpha(W_\psi) - 1\right] + h_\alpha\left(\frac{p}{1+4\epsilon'}\right)}{H_\alpha(W_\rho) - 1}, \quad (\text{if } H_\alpha(W_\rho) > 1),$$

$$n \geq \frac{k\left[1 - H_\alpha(W_\psi)\right] - h_\alpha\left(\frac{p}{1+4\epsilon'}\right)}{\left[1 - H_\alpha(W_\rho)\right]}, \quad (\text{if } H_\alpha(W_\rho) < 1).$$

Note: *extends to odd d for generic stabilizer codes for* $1 \mapsto \log d$

Lower bound comparison

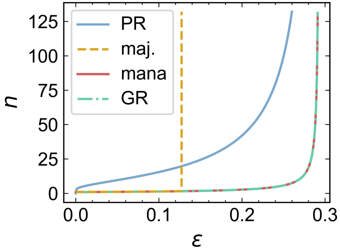
Task: *Distill perfect* $|H\rangle$ *up to* $\epsilon' = 10^{-9}$ *from* $\rho(\epsilon)^{\otimes n}$: $\rho(\epsilon) := (1-\epsilon)\,|H\rangle\langle H| + \epsilon\frac{\mathbb{1}}{2}$

<u>Monotone bounds:</u>

PR = projective robustness[12]

GR = generalized robustness[13]

mana[14]



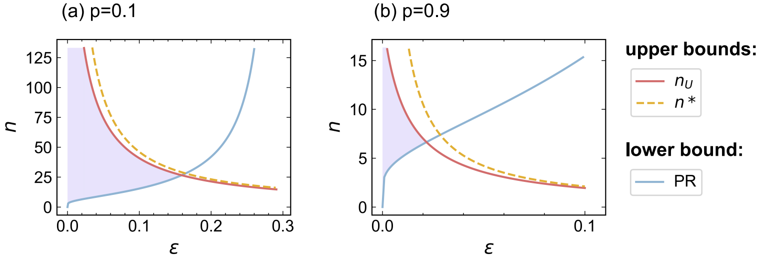
[14] Regula 2022.

[14] Seddon et al. 2021.

[14] Veitch, Mousavian, et al. 2014.

Upper bounds

Task: *Distill perfect $|H\rangle$ up to output error $\epsilon' = 10^{-9}$, from $\rho(\epsilon)^{\otimes n}$*



PR = projective robustness[15]

$$(\alpha\text{=2 bound}) \rightarrow n^* := 2\frac{\log p - \log[1+4\epsilon']}{\log[1-\epsilon+\frac{\epsilon^2}{2}]}$$

[15] Regula (2022)

Future work

Can we extend to... CV systems, full set of qubit SO, non-rebit magic states?

Recall: *for non-rebit magic states (e.g. $T|+\rangle$) our rep W_ρ is complex-valued*

But: can always define valid quasiprobability distributions from

$$\boldsymbol{w}_\rho := \mathrm{Re}\{W_\rho\} \oplus \mathrm{Im}\{W_\rho\},$$
$$\boldsymbol{r}_\tau := \frac{1}{2} W_\tau \oplus W_\tau,$$

such that:

$$\exists\, \mathcal{E} \in \mathcal{O}_{CSS} : \quad \mathcal{E}(\rho) = \rho' \text{ and } \mathcal{E}(\tau) = \tau' \quad \implies \quad D_\alpha(\boldsymbol{w}_\rho || \boldsymbol{r}_\tau) \geq D_\alpha(\boldsymbol{w}_{\rho'} || \boldsymbol{r}_{\tau'}).$$

“complex relative majorization” :O

Thanks! Questions?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